Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals

TRENT UNIVERSITY, Winter 2021

Solutions to Quiz #1

Tuesday, 19 January.

Do *both* of the following questions. Show all your work!

1. Use the ε–δ definition of limits to verify that $\lim_{x\to 3}(1-2x) = -5$. *[2.5]*

SOLUTION. We need to show that given any $\varepsilon > 0$, there is a $\delta > 0$ such that if $|x-3| < \delta$, then $|(1-2x)-(-5)| < \varepsilon$.

Suppose we are given an $\varepsilon > 0$. We will reverse-engineer the corresponding δ:

$$\begin{aligned} |(1-2x)-(-5)| < \varepsilon &\iff |1-2x+5| < \varepsilon \\ &\iff |-2x+6| < \varepsilon \\ &\iff |-2\,(x-3)| < \varepsilon \\ &\iff |-2| \cdot |x-3| < \varepsilon \\ &\iff 2|x-3| < \varepsilon \\ &\iff |x-3| < \frac{\varepsilon}{2} \end{aligned}$$

Now let $\delta = \dfrac{\varepsilon}{2}$. Since every step in the above chain is reversible, it follows that if $|x-3| < \delta = \dfrac{\varepsilon}{2}$, then $|(1-2x)-(-5)| < \varepsilon$, as required.

Since we can find a suitable $\delta > 0$ for any given $\varepsilon > 0$, it follows by the ε–δ definition of limits that $\lim_{x\to 3}(1-2x) = -5$. $\square$

2. Use the ε–δ definition of limits to verify that $\lim_{x\to 1}(x-1)^2 = 0$. *[2.5]*

SOLUTION. We need to show that given any $\varepsilon > 0$, there is a $\delta > 0$ such that if $|x-1| < \delta$, then $\left|(x-1)^2 - 0\right| < \varepsilon$.

Suppose we are given an $\varepsilon > 0$. We will reverse-engineer the corresponding δ:

$$\begin{aligned} \left|(x-1)^2 - 0\right| < \varepsilon &\iff \left|(x-1)^2\right| < \varepsilon \\ &\iff |x-1|^2 < \varepsilon \\ &\iff |x-1| = \sqrt{|x-1|^2} < \sqrt{\varepsilon} \end{aligned}$$

Now let $\delta = \sqrt{\varepsilon}$. Since every step in the above chain is reversible, it follows that if $|x-3| < \delta = \sqrt{\varepsilon}$, then $\left|(x-1)^2 - 0\right| < \varepsilon$, as required.

Since we can find a suitable $\delta > 0$ for any given $\varepsilon > 0$, it follows by the ε–δ definition of limits that $\lim_{x\to 1}(x-1)^2 = 0$.

Note that this is one of the rare situations where we apply the ε–δ definition of limits to verify that the limit of a quadratic function is correct and do not have to play estimation games to find a suitable δ. $\blacksquare$