Two more tests for convergence (Section 11.7)

The following tests are particularly useful when handling powerseries (next section!). First, the **Ratio Test**:

Suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$. If

1) $L<1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely;
2) $L>1$, then $\sum_{n=0}^{\infty} a_n$ diverges; and
3) $L=1$, then the test tells you nothing. (That is, the series might converge absolutely, or it might converge conditionally, or it might diverge.)

The Ratio Test tends to work especially well for series where the individual terms are built using multiplication and/or division. (Excepting things like n^n; the Ratio Test can be made to work, but the algebra is painful.)

For example, consider $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$. Since

$$L = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} \cdot \frac{(n+1)^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{1}{2} \cdot \frac{n^2+2n+1}{n^2} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n^2}{n^2} + \frac{2n}{n^2} + \frac{1}{n^2}\right)$$

$$= \frac{1}{2} \lim_{n\to\infty} \left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{1}{2}(1+0+0) = \frac{1}{2} < 1,$$

the series $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$ converges absolutely.

For another example, consider $\sum_{n=0}^{\infty} \frac{(-1)^n n!}{3 \cdot 6^n}$.

Since $L = \lim_{n\to\infty} \left| \dfrac{\frac{(-1)^{n+1}(n+1)!}{3 \cdot 6^{n+1}}}{\frac{(-1)^n n!}{3 \cdot 6^n}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)!}{3 \cdot 6^{n+1}} \cdot \frac{3 \cdot 6^n}{(-1)^n n!} \right|$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{(n+1)!}{n!} \cdot \frac{\cancel{3} \cdot 6^n}{\cancel{3} \cdot 6^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| (-1) \cdot (n+1) \cdot \frac{1}{6} \right| = \lim_{n\to\infty} \frac{n+1}{6} = \infty > 1,$$

the series $\sum_{n=0}^{\infty} \frac{(-1)^n n!}{3 \cdot 6^n}$ diverges.

For a third example, consider our old friend, the alternating harmonic series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

Since $L = \lim_{n\to\infty} \left| \dfrac{\frac{(-1)^{n+1+1}}{n+1}}{\frac{(-1)^{n+1}}{n}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+2}}{n+1} \cdot \frac{n}{(-1)^{n+1}} \right|$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+2}}{(-1)^{n+1}} \cdot \frac{n}{n+1} \right| = \lim_{n\to\infty} \left| (-1) \cdot \frac{n}{n+1} \right|$$

$$= \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{n}{n} + \frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1,$$

the Ratio Test tells us nothing about whether the series converges or not. [We already know it converges by the Alternating Series Test, but only conditionally because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.]

For a proof that the Ratio Test works, see §11.7 in the textbook. (Basically, if $L>1$ or if $L<1$, you can compare the series to a suitable geometric series.)

The Ratio Test will be the first test we try when dealing with power series; the Root Test is a backup if the Ratio Test is too messy to compute the limit for. (The other tests we know we need to handle the times when $L=1$.)

Here's the <u>Root Test</u>:

Suppose that $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = L$.

If 1) $L<1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely;

2) $L>1$, then $\sum_{n=0}^{\infty} a_n$ diverges; and

3) $L=1$, then the test tells you nothing.

The downside to the Root Test is that it usually results in difficult algebra. For example, have fun applying it to $\sum_{n=0}^{\infty} \frac{5^n+n^2}{2^n+n!}$ ... It tends to work pretty well, though, for series in which $|a_n|$ is essentially an n^{th} power. For example, consider the series $\sum_{n=0}^{\infty} \frac{(-1)^n 5^n}{n^n} = \sum_{n=0}^{\infty} \left(\frac{-5}{n}\right)^n$.

Since

$$L = \lim_{n\to\infty} \left|\left(\frac{-5}{n}\right)^n\right|^{1/n} = \lim_{n\to\infty} \left[\left(\frac{5}{n}\right)^n\right]^{1/n}$$

$$= \lim_{n\to\infty} \left(\frac{5}{n}\right)^{n/n} = \lim_{n\to\infty} \frac{5}{n} = 0 < 1,$$

it follows by the Root Test that the given series converges absolutely. Try this – just to see! – with the Ratio Test...

Since the textbook skips it, here is part of the proof that the Root Test works:

Suppose $\lim_{n\to\infty} |a_n|^{1/n} = L < 1$. Choose an r with $L < r < 1$; then, for some $k > 0$, we have $|a_n|^{1/n} < r$ for all $n \geq k$. (Think about how limits as $n \to \infty$ work...) But if $|a_n|^{1/n} < r$, then $|a_n| < r^n$, so $\sum_{n=k}^{\infty} |a_n|$ converges by comparison with the geometric series $\sum_{n=k}^{\infty} r^n$, which converges because $0 \leq r < 1$. But if $\sum_{n=k}^{\infty} |a_n|$ converges, so does $\sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{k-1} |a_n| + \sum_{n=k}^{\infty} |a_n|$, which means that $\sum_{n=0}^{\infty} a_n$ converges absolutely.

(The case with $L > 1$ works similarly, with an $L > r > 1$, to get divergence.)