Alternating Series Test (Section 11.3 of the text)

An alternating series is one which has each positive term followed by a negative one and vice versa. For example, the alternating harmonic series is obtained from the harmonic series by subtracting every second term instead of adding it:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots$$

It turns out that alternating series may converge even if their purely positive (or negative) counterparts do not. For example, the alternating harmonic series converges, even though the harmonic series diverges. The following test tells us when an alternating series converges.

Alternating Series Test

Suppose $\sum_{n=0}^{\infty} a_n$ is an alternating series. If

(1) $\lim_{n\to\infty} |a_n| = 0$

& (2) Past some point the underlying sequence is non-increasing, ie $|a_{n+1}| \leq |a_n|$ for all n past some point,

then $\sum_{n=0}^{\infty} a_n$ converges.

Check out the proof and the preceding discussion in the text.

The key takeaway from that discussion and proof is that if $\sum_{n=0}^{\infty} a_n$ is an alternating series that converges to some value A, then one can easily estimate how close the partial sums of the series are to A, namely that

$$\left| \left(\sum_{n=0}^{k} a_n \right) - A \right| \leq |a_{k+1}|.$$

This little fact can come in handy in Assignment #5 and even more so in the new, Maple-less, alternative, Assignment #5.1.

Def'n: A series which converges, call it $\sum_{n=0}^{\infty} a_n$, but for which the corresponding series of positive terms $\sum_{n=0}^{\infty} |a_n|$ diverges, is said to be conditionally convergent. If $\sum_{n=0}^{\infty} |a_n|$ converges, then $\sum_{n=0}^{\infty} a_n$ is said to be absolutely convergent.

(Section 11.5 of the text.)

For example, the alternating harmonic series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, is conditionally convergent because the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, diverges.

On the other hand, the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots$$

is absolutely convergent because the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots \quad \text{converges.}$$

[By the p-Test, since $p = 2 > 1$.]

Note that this provides one with a useful shortcut in some cases. There is no need to check that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is convergent by the Alternating Series Test, because it is easier to check that it is absolutely convergent by using the p-Test to show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

The key is that an absolutely convergent series must be convergent.

<u>Example:</u> Consider the alternating series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n[\ln(n)]^2}.$$ This converges

by the Alternating Series Test:

(0) $n[\ln(n)]^2 > 0$ for all $n \geq 2$ and $(-1)^n$ alternates, so the series alternates.

(1) $$\lim_{n\to\infty} \left| \frac{(-1)^n}{n[\ln(n)]^2} \right| = \lim_{n\to\infty} \frac{1}{n[\ln(n)]^2} = 0$$

since $1 \to 1$ and $n[\ln(n)]^2 \to \infty$ as $n \to \infty$.

(2) Since $n[\ln(n)]^2$ is increasing as n increases, since n, $\ln(n)$, and $[\ \ ]^2$ are all increasing functions, we have that

$$|a_{n+1}| = \frac{1}{(n+1)[\ln(n+1)]^2} < \frac{1}{n[\ln(n)]} = |a_n|$$

for all $n \geq 2$.

It follows that $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n[\ln(n)]^2}$ converges by the alternating series test.

On the other hand, it's pretty easy to show that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n[\ln(n)]^2}$ converges by the Integral Test:

$$\int_2^{\infty} \frac{1}{x[\ln(x)]^2}\,dx = \lim_{a\to\infty} \int_2^{a} \frac{1}{x[\ln(x)]^2}\,dx$$

We'll now substitute $u = \ln(x)$, so $du = \frac{1}{x}dx$.

$$= \lim_{a\to\infty} \int_{x=2}^{x=a} \frac{1}{u^2}\,du = \lim_{a\to\infty} \int_{x=2}^{x=a} u^{-2}\,du$$

$$= \lim_{a\to\infty} \frac{u^{-1}}{-1}\Big|_{x=2}^{x=a} = \lim_{a\to\infty} \frac{-1}{u}\Big|_{x=2}^{x=a}$$

$$= \lim_{a\to\infty} \frac{-1}{\ln(x)}\Big|_2^a = \lim_{a\to\infty} \left[\frac{-1}{\ln(a)} - \frac{-1}{\ln(2)}\right]$$

$$= \frac{1}{\ln(2)} \text{ since } \frac{1}{\ln(a)} \to 0 \text{ as } a \to \infty.$$

It follows that $\sum_{n=2}^{\infty} \frac{1}{n[\ln(n)]^2}$ is convergent by the integral test, which in turn means that

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n[\ln(n)]^2}$$ is absolutely convergent and hence convergent.

Which is the better way here? It depends on what you want to know. In this example, applying the Alternating Series Test to just get convergence was probably easier than applying the Integral Test to get absolute convergence. On the other hand, knowing that a series is absolutely convergent means it can be used (or abused) in ways a conditionally convergent series cannot. The most spectacular example is the following facts:

- No matter how you rearrange or scramble ~~the~~ terms of an absolutely convergent series, you always get the same sum.
- On the other hand, a conditionally convergent can be rearranged to get any sum you want.

We'll take a look at an example of the latter fact. Next time! ☺