Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2020

Solutions to Assignment #6
Gregory's Series

The rate of convergence of the alternating series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$ got a little attention on Assignment #5 and its *Maple*-less alternative, Assignment #5.1. It was mentioned there that this series adds up to $\frac{\pi}{4}$. In this assignment we shall see why it does so, using a method similar to that used in the lecture notes for 2020-03-19 to show that the alternating harmonic series adds up to $\ln(2)$ if it is not rearranged.

1. Find a power series representation of $\arctan(x)$. *[6]*

Hint: Start with the fact that $\displaystyle\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$.

SOLUTION. Recall that a geometric series with first term a and common ratio r, $\displaystyle\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$, has a sum of $\dfrac{a}{1-r}$ as long as $|r| < 1$. If we take $a = 1$ and $r = -x^2$, this means that

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 + \cdots = \sum_{n=0}^{\infty}(-1)^n x^{2n},$$

at least as long as $\left|-x^2\right| < 1$, *i.e.* $|x| < 1$. It now follows that

$$\begin{aligned}\arctan(x) &= \int \frac{1}{1+x^2}\,dx = \int \left(1 - x^2 + x^4 - x^6 + x^8 - \cdots\right)dx \\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.\end{aligned}$$

Plugging in $x = 0$ at both ends gives us $0 = \arctan(0) = C + 0$, so $C = 0$. Thus $\displaystyle\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, at least when the series converges. ■

The power series representation obtained here is usually called *Gregory's series* after the Scottish mathematician and astronomer James Gregory (1638-1675) who rediscovered it in 1668. It had previously been discovered by the Indian mathematician and astronomer Madhava of Sangamagrama (1350-1410), and was independently rediscovered in 1676 by Gottfried Leibniz (1646-1716), one of the co-inventors of calculus.

2. Use the power series representation of $\arctan(x)$ obtained in **1** to conclude that $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}$. *[1]*

SOLUTION. When we plug $x = 1$ into $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \arctan(x)$, we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \arctan(1) = \frac{\pi}{4}\,.$$

The only point to really quibble over here is whether the series converges at all; those who are concerned about that can check that it does using the Alternating Series Test. ■

3. How practical is it to use Gregory's series to compute (approximations to) π? *[3]*

SOLUTION. As noted in the solutions to Assignment #5.1, you need several thousand terms of the case $x = 1$ of Gregory's series to guarantee that the partial sums are within 0.001 of $\frac{\pi}{4}$. This suggests that it is not an efficient way to compute approximations to $\frac{\pi}{4}$, and hence (by multiplying by 4 :-) to π. This is indeed the case. The other series mentioned in Assignment #5.1 that adds up to $\frac{\pi}{4}$ converges much faster, albeit at some cost in terms of the individual terms being a little more complicayed to work out, and is much more useful if you actually want to compute π. Various other series and other techniques have been worked out that are more practical yet.

On the other hand, Gregory's series was an improvement over some previous, even more inefficient, techniques for computing π, and a number of the more efficient series for the job, starting with the one given in Assignment #5.1, are related to it algebraically or via identities among inverse trigonometric functions. It has, therefore, some significance historically and as a basis for devising better techniques.

You can read up about π, its history, and techniques for computing it, in the Wikipedia article on π at `https://en.wikipedia.org/wiki/Pi` . ■