Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2020

Solution to Assignment #3
Plane Pursuit

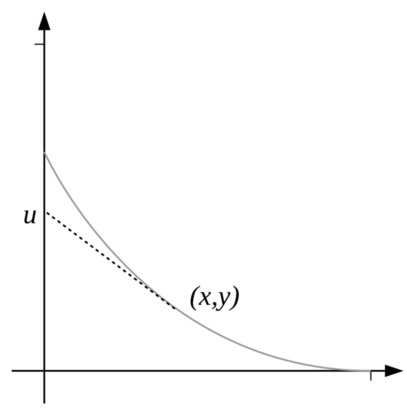
A straight train track running north-south on a flat and otherwise featureless plain* meets a straight road running east-west at right angles. As the last car of a train heading north at 100 km/h passes this intersection, a drone† is flying west directly over the road at 200 km/h is 1 km east of the intersection. At this instant, the drone's controller decides to have it chase the train and thereafter keeps the drone headed directly towards the last car of the train until it catches up.

1. If the train and drone maintain their speeds of 100 km/h and 200 km/h, how far from the crossing does the drone catch up with the last car of the train? *[10]*

Hint: Find a differential equation describing the drone's path and take it from there. Suppose you set things up so that the x-axis runs along the road and the y-axis along the train tracks, with the origin at the intersection, all scaled in kilometres. If the path followed by the drone is the graph of $y = f(x)$, then $f(1) = 0$ with $\left.\dfrac{dy}{dx}\right|_{x=1} = f'(1) = 0$. The differential equation ought to be something like $2x\dfrac{d^2y}{dx^2} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$ – it will be part of your job to explain this. When solving for $y = f(x)$, it will probably be a good idea to solve for $\dfrac{dy}{dx} = f'(x)$ first.

SOLUTION. The solution breaks down into two parts: getting to the differential equation mentioned in the hint and then solving it.

For the first part, consider the following diagram of the situation of the instant that the drone is at the point (x, y).



* The plain is plainly a plain plane and a plane plain.

† Is the drone a plain plane flying over the plane plain?

Note that the train is moving on the y-axis and the the drone is moving directly towards the end of the train. So the end of the train must be at the point $(0, u)$ where the tangent line to the curve traced out by the drone intercepts the y-axis. The tangent line to the curve, at the point (x, y), has slope $\frac{dy}{dx}$. You can check for yourselves that the line with slope m passing through the point (c, d) intercepts the y-axis at the point $(0, d - cm)$. It follows that in our case

$$u = y - x\frac{dy}{dx}.$$

Note that at the instant the drone is at (x, y), the train has travelled a distance of u *km*, *i.e.* from $(0, 0)$ to $(0, u)$, since the instant that the drone took off in pursuit. In the same period, the drone has travelled a distance equal to the arc length of the curve $y = f(x)$ traced by the drone from x to 1. Plugging that into the appropriate integral formula gives

$$\text{arc length} = \int_x^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx\,,$$

with a slight abuse of notation. (Namely?) Since the drone is moving at twice the speed of the train, it covers twice the distance in a given period of time, so $2u$ is equal to the arc length above. This gives us the differential/integral equation:

$$2\left(y - x\frac{dy}{dx}\right) = \int_x^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Differentiating both sides gives

$$2\frac{dy}{dx} - 2 \cdot 1 \cdot \frac{dy}{dx} - 2x\frac{d^2y}{dx^2} = -\sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

which, after simplifying, amounts to the desired differential equation:

$$2x\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

For the second part, we need to solve this differential equation. *Per* the hint, we get to assume that when $x = 1$, $y = f(1) = 0$ and $\left.\frac{dy}{dx}\right|_{x=1} = f'(1) = 0$. Solving second-order differential equations (those involving second derivatives) is usually much harder than solving first-order differential equations.

i. Solving it by hand: In this case, however, we can get around this problem by solving for $\frac{dy}{dx}$ — remember that $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ — and then getting y by integrating $\frac{dy}{dx}$.

If we set $z = \frac{dy}{dx}$, then $\frac{dz}{dx} = \frac{d^2y}{dx^2}$, so we can rewrite our differential equation as

$$2x\frac{dz}{dx} = \sqrt{1 + z^2}\,.$$

This is a "separable" first-order differential equation, which there is a recipe for. Abusing notation a bit (how?), rearrange the equation to group the bits involving z and the bits involving x on opposite sides of the equal sign:

$$\frac{dz}{\sqrt{1+z^2}} = \frac{dx}{2x},$$

Now integrate each side (with respect to z and x, respectively) to get:

$$\ln\left(z+\sqrt{1+z^2}\right) = \ln\left(\sqrt{x}\right) + C$$

where C is some constant. Exponentiating both sides now gives

$$z+\sqrt{1+z^2} = e^{\ln\left(z+\sqrt{1+z^2}\right)} = e^{\ln\left(\sqrt{x}\right)+C} = K\sqrt{x},$$

where $K = e^C$. Since $z = \frac{dy}{dx} = 0$ when $x = 1$, it follows that $K = 1$.

We now have to solve $z+\sqrt{1+z^2} = \sqrt{x}$ for z in terms of x. Squaring both sides gives

$$z^2 + 2z\sqrt{1+z^2} + \left(1+z^2\right) = x,$$

which unfortunately still involves $\sqrt{1+z^2}$. We get around this by rearranging the equation to get

$$2z\sqrt{1+z^2} = x - 2z^2 - 1,$$

and then squaring to get

$$4z^2\left(1+z^2\right) = 4z^2 + 4z^4 = x^2 - 4xz^2 - 2x + 4z^4 + 4z^2 + 1,$$

which simplifies to

$$0 = x^2 - 4xz^2 - 2x + 1.$$

Rearranging now gives

$$4xz^2 = x^2 - 2x + 1 = (x-1)^2,$$

from which it follows that

$$\frac{dy}{dx} = f'(x) = z = \frac{x-1}{2\sqrt{x}} = \frac{\sqrt{x}}{2} - \frac{1}{2\sqrt{x}}.$$

[Whew!]

We still have to integrate $z = \frac{dy}{dx} = f'(x)$ to get $y = f(x)$ as a function of x:

$$y = \int\left(\frac{\sqrt{x}}{2} - \frac{1}{2\sqrt{x}}\right)dx = \frac{x^{3/2}}{3} - \sqrt{x} + C$$

C is some constant which we can pin down because $y = 0$ when $x = 1$, so $C = \dfrac{2}{3}$, and thus

$$y = f(x) = \frac{x^{3/2}}{3} - \sqrt{x} + \frac{2}{3}\,.$$

We can now answer the question. When $x = 0$, $y = f(0) = \frac{2}{3}$. Thus the drone catches up with the train $\frac{2}{3}$ *km* from the crossing. □

ii. Using Sagemath: Most of you will have seen and used `Maple`; just to be different, we will use a freeware alternative, the open source software suite `Sagemath`. In particular, to avoid having to install Sagemath, we will use the portion of `Sagemath`'s capabilities available online at: `https://sagecell.sagemath.org`

Typing in

```
x = var('x'); y = function(('y')(x)
DE = 2*x*diff(y,x,2) == sqrt(1+(diff(y,x,1))^2)
desolve(DE,y).expand()
```

and hitting the "Evaluate" button gives the output:

```
_K2 + 1/3*e^(1/2*_K1 + 3/2*log(x)) - e^(-1/2*_K1 + 1/2*log(x))
```

That is, according to `Sagemath` the general solution to the given differential equation is

$$y = K_2 + \frac{1}{3}e^{\frac{1}{2}K_1+\frac{3}{2}\ln(x)} - e^{-\frac{1}{2}K_1+\frac{1}{2}\ln(x)}\,,$$

where K_1 and K_2 are unknown constants, which we will shortly pin down using the initial conditions given in the problem. Before we do so, though, we'll try to simplify the somewhat cumbersome expression above:

$$\begin{aligned} y &= K_2 + \frac{1}{3}e^{\frac{1}{2}K_1+\frac{3}{2}\ln(x)} - e^{-\frac{1}{2}K_1+\frac{1}{2}\ln(x)} \\ &= \frac{1}{3}e^{K_1/2}e^{\ln\left(x^{3/2}\right)} - e^{-K_1/2}e^{\ln\left(x^{1/2}\right)} + K_2 \\ &= \frac{1}{3}e^{K_1/2}x^{3/2} - e^{-K_1/2}x^{1/2} + K_2 \end{aligned}$$

We will also need to know its derivative:

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{3}e^{K_1/2}x^{3/2} - e^{-K_1/2}x^{1/2} + K_2\right] = \frac{1}{2}e^{K_1/2}x^{1/2} - \frac{1}{2}e^{-K_1/2}x^{-1/2}$$

Now, since $\dfrac{dy}{dx} = 0$ at $x = 1$, we have that $\frac{1}{2}e^{K_1/2} - \frac{1}{2}e^{-K_1/2} = 0$, which boils down to $e^{K_1/2} = e^{-K_1/2}$, which in turn requires that $K_1 = -K_1$, and so $K_1 = 0$. Thus $y = \dfrac{1}{3}x^{3/2} - x^{1/2} + K_2$. Since $y = 0$ when $x = 1$, we now have $\dfrac{1}{3} - 1 + K_2 = 0$, and so $K_2 = \dfrac{2}{3}$.

Thus $y = \dfrac{x^{3/2}}{3} - \sqrt{x} + \dfrac{2}{3}$; plugging in $x = 0$ tells us that the drone catches up with the train $\frac{2}{3}$ *km* from the crossing. ■