Mathematics 1120H – Calculus II: Integrals and Series

TRENT UNIVERSITY, Winter 2020

Assignment #1
Gamma Function

Due on Thursday, 16 January.

One of the big uses of integrals in various parts of mathematics is to define functions that are otherwise difficult to nail down. For example, consider the factorial function on the non-negative integers, defined by $0! = 1$ and $(n+1)! = n! \cdot (n+1)$. (It's pretty easy to check that if $n \geq 1$ is an integer, then $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$.) The factorial function turns up in many parts of mathematics, including algebra, calculus [wait till we do series!], combinatorics, and number theory. The essentially discrete factorial function has a continuous (also differential and integrable) counterpart, which also comes up a fair bit in both applied and theoretical mathematics, namely the gamma function $\Gamma(x)$. This can be defined in a number of different ways, but the most common combines a limit and an integral:

$$\Gamma(x) = \lim_{a \to \infty} \int_0^a t^{x-1} e^{-t} \, dt$$

This definition makes sense for all $x > 0$. The expression $\displaystyle\lim_{a \to \infty} \int_0^a t^{x-1} e^{-t} \, dt$ is usually abbreviated a little as $\displaystyle\int_0^\infty t^{x-1} e^{-t} \, dt$, something we'll see more of when we do "improper integrals" (§9.7 in the textbook).

1. Verify that $\Gamma(1) = 1$. *[2]*

2. Show that $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$. *[2]*

3. Use **1** and **2** to show that $\Gamma(n+1) = n!$ for every integer $n \geq 0$. *[2]*

4. Use the limit definition of the derivative and the improper integral definition of $\Gamma(x)$ to find an integral definition of its derivative, $\Gamma'(x)$. *[4]*