Mathematics 1120H – Calculus II: Integrals and Series

Trent University, Winter 2019

Solutions to the Quizzes

Quiz #1. Wednesday, 20 June. *[10 minutes]*

Compute each of the following integrals.

1. $\displaystyle\int_0^{\pi/2} \cos(x)\sqrt{\sin(x)}\,dx$ *[3]* 2. $\displaystyle\int \frac{1}{x\ln(x)}\,dx$ *[2]*

Solutions. 1. We will use the substitution $u = \sin(x)$, so $\dfrac{du}{dx} = \cos(x)$ and hence $du = \cos(x)\,dx$, and change the limits as we go along: $\begin{matrix} x & 0 & \pi/2 \\ u & 0 & 1 \end{matrix}$, since $\sin(0) = 0$ and $\sin(\pi/2) = 1$.

$$\int_0^{\pi/2} \cos(x)\sqrt{\sin(x)}\,dx = \int_0^1 \sqrt{u}\,du = \int_0^1 u^{1/2}\,du = \left.\frac{u^{3/2}}{3/2}\right|_0^1 = \left.\frac{2}{3}u^{3/2}\right|_0^1$$
$$= \frac{2}{3}\cdot 1^{3/2} - \frac{2}{3}\cdot 0^{3/2} = \frac{2}{3} - 0 = \frac{2}{3} \quad \square$$

2. We will use the substitution $w = \ln(x)$, so $\dfrac{dw}{dx} = \dfrac{1}{x}$ and hence $dw = \dfrac{1}{x}\,dx$.

$$\int \frac{1}{x\ln(x)}\,dx = \int \frac{1}{w}\,dw = \ln(w) + C = \ln(\ln(x)) + C \quad \blacksquare$$

Quiz #2. Friday, 25 January. *[10 minutes]*

1. Compute $\displaystyle\int_{-1}^0 x^2 e^{x+1}\,dx$. *[5]*

Solution. We will use integration by parts twice, the first time with $u = x^2$ and $v' = e^{x+1}$, so that $u' = 2x$ and $v = \int e^{x+1}\,dx = \int e \cdot e^x\,dx = e \cdot e^x = e^{x+1}$.

$$\int_{-1}^0 x^2 e^{x+1}\,dx = \left. x^2 e^{x+1}\right|_{-1}^0 - \int_{-1}^0 2xe^{x+1}\,dx = 0^2 e^{0+1} - (-1)^2 e^{-1+1} - 2\int_{-1}^0 xe^{x+1}\,dx$$

We use parts again with $a = x$ and $b' = e^{x+1}$, so $a' = 1$ and $b = e^{x+1}$.

$$= 0 - 1\cdot e^0 - 2\left[\left. xe^{x+1}\right|_{-1}^0 - \int_{-1}^0 1e^{x+1}\,dx\right]$$
$$= -1\cdot 1 - 2\left[0e^{0+1} - (-1)e^{-1+1} - \left. e^{x+1}\right|_{-1}^0\right]$$
$$= -1 - 2\left[0 - (-1)e^0 - \left(e^{0+1} - e^{-1+1}\right)\right] = -1 - 2\left[1\cdot 1 - \left(e^1 - e^0\right)\right]$$
$$= -1 - 2\left[1 - (e-1)\right] = -1 - 2\left[2 - e\right] = -1 - 2\cdot 2 + 2e = 2e - 5 \quad \blacksquare$$

Quiz #3. Friday, ~~32 January~~ 1 February. *[12 minutes]*

Compute each of the following integrals.

1. $\displaystyle\int_0^{\pi/4} \tan^2(x)\, dx$ *[2.5]*   2. $\displaystyle\int_0^{\pi/2} \cos^3(x)\sin^2(x)\, dx$ *[2.5]*

SOLUTIONS. 1. *(Trig identity)* We will use the trigonometric identity $\tan^2(x) = \sec^2(x) - 1$.

$$\begin{aligned}\int_0^{\pi/4} \tan^2(x)\, dx &= \int_0^{\pi/4} \left(\sec^2(x) - 1\right)\, dx = \left(\tan(x) - x\right)\Big|_0^{\pi/4} \\ &= \left(\tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\right) - (\tan(0) - 0) = \left(1 - \frac{\pi}{4}\right) - (0 - 0) = 1 - \frac{\pi}{4} \quad \Box\end{aligned}$$

1. *(Reduction formula)* We will apply the reduction formula

$$\int \tan^k(x)\, dx = \frac{1}{k-1}\tan^{k-1}(x) - \int \tan^{k-2}(x)\, dx\,.$$

Here goes:

$$\begin{aligned}\int_0^{\pi/4} \tan^2(x)\, dx &= \frac{1}{2-1}\tan^{2-1}(x)\Big|_0^{\pi/4} - \int_0^{\pi/4} \tan^{2-2}(x)\, dx \\ &= \tan(x)\Big|_0^{\pi/4} - \int_0^{\pi/4} \tan^0(x)\, dx = \left[\tan\left(\frac{\pi}{4}\right) - \tan(0)\right] - \int_0^{\pi/4} 1\, dx \\ &= [1 - 0] - x\Big|_0^{\pi/4} = 1 - \left[\frac{\pi}{4} - 0\right] = 1 - \frac{\pi}{4} \quad \Box\end{aligned}$$

2. *(Trig identity and substitution)* We will use the trigonometric identityt $\cos^2(x) = 1 - \sin^2(x)$ and then substitute $u = \sin(x)$, so $du = \cos(x)\, dx$, and change limits as we go along: $\begin{matrix} x & 0 & \pi/2 \\ u & 0 & 1 \end{matrix}$.

$$\begin{aligned}\int_0^{\pi/2} \cos^3(x)\sin^2(x)\, dx &= \int_0^{\pi/2} \cos^2(x)\cos(x)\sin^2(x)\, dx \\ &= \int_0^{\pi/2} \left(1 - \sin^2(x)\right)\sin^2(x)\cos(x)\, dx = \int_0^1 \left(1 - u^2\right)u^2\, du \\ &= \int_0^1 \left(u^2 - u^4\right)\, du = \left(\frac{u^3}{3} - \frac{u^5}{5}\right)\Big|_0^1 \\ &= \left(\frac{1^3}{3} - \frac{1^5}{5}\right) - \left(\frac{0^3}{3} - \frac{0^5}{5}\right) = \left(\frac{1}{3} - \frac{1}{5}\right) - (0 - 0) \\ &= \left(\frac{5}{15} - \frac{3}{15}\right) - 0 = \frac{2}{15} \quad \Box\end{aligned}$$

2. *(Trig identity and reduction formula)* We will apply the trig identity $\sin^2 x = 1 - \cos^(x)$ and the reduction formula

$$\int \cos^k(x)\, dx = \frac{1}{k}\cos^{k-1}(x)\sin(x) + \frac{k-1}{k}\int \cos^{k-2}(x)\, dx\,.$$

Here goes:

$$\begin{aligned}
\int_0^{\pi/2} \cos^3(x)\sin^2(x)\,dx &= \int_0^{\pi/2} \cos^3(x)\left(1-\cos^2(x)\right)\,dx = \int_0^{\pi/2} \left(\cos^3(x)-\cos^5(x)\right)\,dx \\
&= \int_0^{\pi/2} \cos^3(x)\,dx - \int_0^{\pi/2} \cos^5(x)\,dx \\
&= \int_0^{\pi/2} \cos^3(x)\,dx - \left[\frac{1}{5}\cos^4(x)\sin(x)\Big|_0^{\pi/2} + \frac{4}{5}\int_0^{\pi/2} \cos^3(x)\,dx\right] \\
&= \int_0^{\pi/2} \cos^3(x)\,dx - \frac{1}{5}\cos^4(x)\sin(x)\Big|_0^{\pi/2} - \frac{4}{5}\int_0^{\pi/2} \cos^3(x)\,dx \\
&= \frac{1}{5}\int_0^{\pi/2} \cos^3(x)\,dx - \frac{1}{5}\cos^4(x)\sin(x)\Big|_0^{\pi/2} \\
&= \frac{1}{5}\left[\frac{1}{3}\cos^2(x)\sin(x)\Big|_0^{\pi/2} + \frac{2}{3}\int_0^{\pi/2} \cos(x)\,dx\right] \\
&\quad - \frac{1}{5}\left[\cos^4\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) - \cos^4(0)\sin(0)\right] \\
&= \frac{1}{5}\left[\frac{1}{3}\cos^2(x)\sin(x)\Big|_0^{\pi/2} + \frac{2}{3}\sin(x)\Big|_0^{\pi/2}\right] - \frac{1}{5}\left[0^4\cdot 1 - 1^4\cdot 0\right] \\
&= \frac{1}{5}\left[\frac{1}{3}\left(\cos^2\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) - \cos^2(0)\sin(0)\right)\right. \\
&\quad \left. + \frac{2}{3}\left(\sin\left(\frac{\pi}{2}\right) - \sin(0)\right)\right] - \frac{1}{5}\cdot 0 \\
&= \frac{1}{5}\left[\frac{1}{3}\left(0^2\cdot 1 - 1^2\cdot 0\right) + \frac{2}{3}(1-0)\right] = \frac{1}{5}\left[\frac{1}{3}\cdot 0\frac{2}{3}\cdot 1\right] \\
&= \frac{1}{5}\cdot\frac{2}{3} = \frac{2}{15} \quad \blacksquare
\end{aligned}$$

NOTE. The truly gung-ho can work out how to use the reduction formula(s) for mixed powers of $\sin(x)$ and $\cos(x)$ to solve question 2.

Quiz #4. Friday, 8 February. *[10 minutes]*

1. Compute $\displaystyle\int \frac{1}{\sqrt{4x^2+8x+8}}\,dx$. *[5]*

SOLUTION. Algebra, substitution, and trig substitution, oh my!

$$\begin{aligned}
\int \frac{1}{\sqrt{4x^2+8x+8}}\,dx &= \int \frac{1}{\sqrt{4\left(x^2+2x+2\right)}}\,dx = \int \frac{1}{2\sqrt{x^2+2x+1+1}}\,dx \\
&= \frac{1}{2}\int \frac{1}{\sqrt{(x+1)^2+1}}\,dx \quad \text{Substitute } u = x+1 \text{, so } du = dx. \\
&= \frac{1}{2}\int \frac{1}{\sqrt{u^2+1}}\,du \quad \text{Substitute } u = \tan(\theta) \text{, so } du = \sec^2(\theta)\,d\theta. \\
&= \frac{1}{2}\int \frac{1}{\sqrt{\tan^2(\theta)+1}}\sec^2(\theta)\,d\theta = \frac{1}{2}\int \frac{\sec^2(\theta)}{\sqrt{\sec^2(\theta)}}\,d\theta \\
&= \frac{1}{2}\int \frac{\sec^2(\theta)}{\sec(\theta)}\,d\theta = \frac{1}{2}\int \sec(\theta)\,d\theta = \frac{1}{2}\ln\left(\tan(\theta)+\sec(\theta)\right)+C \\
&= \frac{1}{2}\ln\left(u+\sqrt{u^2+1}\right)+C = \frac{1}{2}\ln\left((x+1)+\sqrt{(x+1)^2+1}\right)+C
\end{aligned}$$

The truly gung-ho can rewrite the final expression as $\frac{1}{2}\ln\left((x+1)+\sqrt{x^2+2x+2}\right)+C$, or even as $\ln\left(\sqrt{(x+1)+\sqrt{x^2+2x+2}}\right)+C$, but that's probably more trouble than it's worth … ■

Quiz #5. Friday, 15 February. *[17 minutes]*

1. Compute $\displaystyle\int \frac{x^2+x+5}{\left(x^2+4\right)(x+1)}\,dx$. *[5]*

SOLUTION. The integrand is a rational function, so we run through the usual partial fractions checklist:

i. Since the numerator of the integrand has degree two, which is less than the degree three of the denominator, we can proceed directly to factoring the denominator.

ii. The denominator, $\left(x^2+4\right)(x+1)$, comes in at least partially factored form. As $x^2+4 \geq 4 > 0$ for all x, the factor x^2+4 has no roots and hence is an irreducible quadratic, which means that the denominator came fully factored.

iii. The partial fraction decomposition of $\dfrac{x^2+x+5}{\left(x^2+4\right)(x+1)}$ is therefore $\dfrac{Ax+B}{x^2+4}+\dfrac{C}{x+1}$ for some constants A, B, and C. Since

$$\begin{aligned}\frac{x^2+x+5}{\left(x^2+4\right)(x+1)} &= \frac{Ax+B}{x^2+4}+\frac{C}{x+1} = \frac{(Ax+B)(x+1)+C\left(x^2+4\right)}{\left(x^2+4\right)(x+1)}\\ &= \frac{Ax^2+Ax+Bx+B+Cx^2+4C}{\left(x^2+4\right)(x+1)}\\ &= \frac{(A+C)x^2+(A+B)x+(B+4C)}{\left(x^2+4\right)(x+1)}\,,\end{aligned}$$

it follows by comparing coefficients of like powers of x in the numerators at the beginning and the end above that $A+C=1$, $A+B=1$, and $B+4C=5$.

iv. We solve the linear equations obtained in the previous step for A, B, and C. The first two equations tell us that $C=1-A=B$. Plugging this into the third equation gives $B+4C=B+4B=5B=5$, and so $B=1$, from which it now follows that $C=B=1$ and $A=1-B=1-1=0$. Thus $\dfrac{x^2+x+5}{\left(x^2+4\right)(x+1)}=\dfrac{1}{x^2+4}+\dfrac{1}{x+1}$.

v. Finally, we integrate:

$$\int \frac{x^2+x+5}{\left(x^2+4\right)(x+1)}\,dx = \int\left(\frac{1}{x^2+4}+\frac{1}{x+1}\right)dx = \int\frac{1}{x^2+4}\,dx+\int\frac{1}{x+1}\,dx$$

Substitute $x=2u$, so $dx=2\,du$, in the first integral, and $w=x+1$, so $dx=dw$, in the second integral.

$$\begin{aligned}&= \int\frac{1}{(2u)^2+4}\,2\,du+\int\frac{1}{w}\,dw = \int\frac{2}{4u^2+4}\,du+\ln(w)\\ &= \frac{2}{4}\int\frac{1}{u^2+1}\,du+\ln(w) = \frac{1}{2}\arctan(u)+\ln(w)+C\end{aligned}$$

Substituting back, note that $u=\dfrac{x}{2}$ and $w=x+1$.

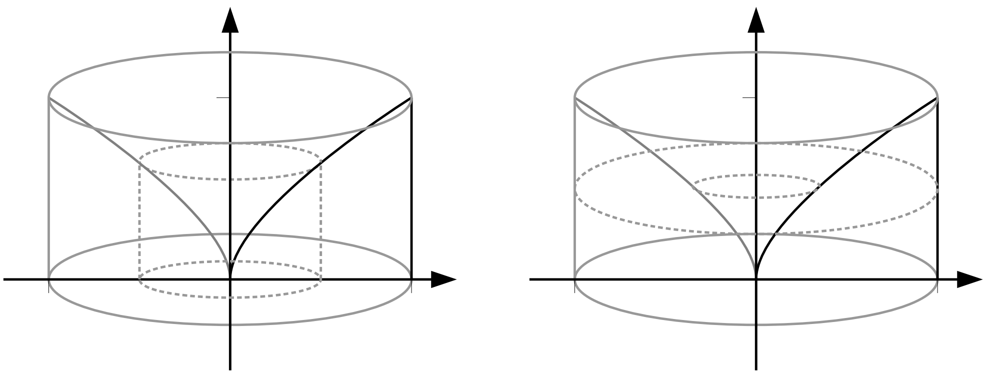
$$= \frac{1}{2}\arctan\left(\frac{x}{2}\right)+\ln(x+1)+C \qquad \blacksquare$$

Quiz #6. Friday, 8 March. *[12 minutes]*

1. Consider the region below $y = \sqrt{x}$ and above $y = 0$ for $0 \le x \le 1$. Find the volume of the solid obtained by revolving this region about the y-axis. *[5]*

SOLUTION. *(Using cylindrical shells.)* Since we are revolving the region about a vertical line, we should use x, the horizontal variiable, if we intend to use the method of cylindrical shells. The cylindrical shell at x has radius equal to the distance from x to the y-axis (*i.e.* $x = 0$), so $r = x - 0 = x$, and height equal to the distance between $y = \sqrt{x}$ and $y = 0$, so $h = \sqrt{x} - 0 = \sqrt{x}$. It follows that the volume of the solid is given by:

$$V = \int_0^1 2\pi r h\,dx = \int_0^1 2\pi x\sqrt{x}\,dx = 2\pi \int_0^1 x^{3/2}\,dx = 2\pi \left.\frac{x^{5/2}}{5/2}\right|_0^1 = \left.\frac{4}{5}\pi x^{5/2}\right|_0^1$$
$$= \frac{4}{5}\pi \cdot 1^{5/2} - \frac{4}{5}\pi \cdot 0^{5/2} = \frac{4}{5}\pi - 0 = \frac{4}{5}\pi \qquad \square$$



(Using washers.) Since we are revolving the region about a vertical line, we should use y, the vertical variiable, if we intend to use the disk/washer method. The washer at y has an outer radius given by the difference between $x = 1$, the right boundary of the original region, and $x = 0$, since the axis of revolution is the y-axis, so $R = 1 - 0 = 1$. This same washer has an inner radius given by the difference between $x = y^2$, since the left boundary of the region is $y = \sqrt{x}$, and $x = 0$, since the axis of revolution is the y-axis, so $r = y^2 - 0 = y^2$. Note also that $0 \le y \le 1$ over the given region. It follows that the volume of the solid is given by:

$$V = \int_0^1 \left(\pi R^2 - \pi r^2\right)\,dy = \pi \int_0^1 \left(1^2 - \left(y^2\right)^2\right)\,dy = \pi \int_0^1 \left(1 - y^4\right)\,dy$$
$$= \pi \left.\left(y - \frac{y^5}{5}\right)\right|_0^1 = \pi\left(1 - \frac{1^5}{5}\right) - \pi\left(0 - \frac{0^5}{5}\right) = \frac{4}{5}\pi - 0 = \frac{4}{5}\pi \qquad \blacksquare$$

Quiz #7. Friday, 15 March. *[15 minutes]*

Determine whether each of the following series converges or diverges.

1. $\displaystyle\sum_{n=0}^{\infty} e^{-n}$ *[2.5]* 2. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{1+n^2}$ *[2.5]*

SOLUTIONS. 1. *(Geometric Series)* $\displaystyle\sum_{n=0}^{\infty} e^{-n} = \sum_{n=0}^{\infty} \frac{1}{e^n} = \sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$ is a geometric series with first term $a = \left(\frac{1}{e}\right)^0 = 1$ and common ratio $r = \frac{1}{e}$. Since $r = \frac{1}{e} < 1$, the series converges; in fact it adds up to $\displaystyle\frac{a}{1-r} = \frac{1}{1-\frac{1}{e}} = \frac{e}{e-1}$. □

1. *(Integral Test)* By the Integral Test, $\displaystyle\sum_{n=0}^{\infty} e^{-n}$ converges or diverges exactly as the improper integral $\displaystyle\int_0^{\infty} e^{-x}\,dx$ does, so we compute the integral. We will use the substitution $u = -x$, so $du = (-1)\,dx$ and thus $dx = (-1)\,du$, and keep the old limits, substituting back in terms of x before using them:

$$\int_0^{\infty} e^{-x}\,dx = \lim_{t\to\infty} \int_0^t e^{-x}\,dx = \lim_{t\to\infty} \int_{x=0}^{x=t} e^u\,(-1)\,du = \lim_{t\to\infty} (-1)e^u\Big|_{x=0}^{x=t}$$
$$= \lim_{t\to\infty} (-1)e^{-x}\Big|_0^t = \lim_{t\to\infty} \left[(-1)e^{-t} - (-1)e^{-0}\right] \lim_{t\to\infty} \left[1 - e^{-t}\right] = 1 - 0 = 1\,,$$

since $e^{-t} \to 0$ as $t \to \infty$. Since the improper integral in question converges to a real number, the given series converges by the Integral Test. □

2. *(Integral Test)* By the Integral Test, $\displaystyle\sum_{n=0}^{\infty} \frac{1}{1+n^2}$ converges or diverges exactly as the improper integral $\int_0^{\infty} \frac{1}{1+x^2}\,dx$ does, so we compute the integral. Recall that $\displaystyle\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$, and that $\arctan(x)$ has a horizontal asymptote of $\frac{\pi}{2}$ as one heads out to infinity.

$$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \arctan(x)\Big|_0^t = \lim_{t\to\infty} [\arctan(t) - \arctan(0)]$$
$$= \lim_{t\to\infty} [\arctan(t) - 0] = \lim_{t\to\infty} \arctan(t) = \frac{\pi}{2}$$

Since the improper integral in question converges to a real number, the given series converges by the Integral Test. □

2. *(Basic Comparison Test and p-Test)* Since $\displaystyle 0 < \frac{1}{1+n^2} < \frac{1}{n^2}$ for all $n \geq 1$, and $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges by the p-Test because $p = 2 > 1$, the Basic Comparison Test tells us that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{1+n^2}$ converges as well. ■

Quiz #8. Friday, 22 March. *[15 minutes]*

Determine whether each of the following series converges or diverges.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$ *[2.5]* 2. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{2^n + 5^n}$ *[2.5]*

SOLUTIONS. 1. *(Alternating Series Test)* First, note that n and $\ln(n)$, and hence also $\frac{1}{n\ln(n)}$, are positive when $n \geq 2$. Since $(-1)^n$ alternates between positive and negative with successive n, it follows that $\frac{(-1)^n}{n\ln(n)}$ alternates as well.

Second, since $n+1 > n$ and $\ln(n+1) > \ln(n)$ (because $\ln(x)$ is an increasing function) for all $n \geq 2$, it follows that $\left|\frac{(-1)^{n+1}}{(n+1)n\ln(n+1)}\right| = \frac{1}{(n+1)n\ln(n+1)} < \frac{1}{n\ln(n)} = \left|\frac{(-1)^n}{n\ln(n)}\right|$ for all $n \geq 2$.

Third, $\displaystyle\lim_{n\to\infty} \left|\frac{(-1)^n}{n\ln(n)}\right| = \lim_{n\to\infty} \frac{1}{n\ln(n)} = 0$ since $n \to \infty$ and $\ln(n) \to \infty$ as $n \to \infty$.

Since the given series satisfies the three conditions of the Alternating Series Test, it converges. □

2. *(Basic Comparison Test)* Observe that the dominant terms in the numerator and denominator are 3^n and 5^n, respectively, which suggests that the given series ought to converge or diverge depending on whether $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{5^n}$ does. The latter series does converge, because it is a geometric series with common ratio $r = \frac{3}{5} < 1$.

For each $n \geq 0$, we have $0 \leq \frac{3^n}{2^n + 5^n} \leq \frac{3^n}{5^n} = \left(\frac{3}{5}\right)^n$ because making the denominator smaller makes the fraction bigger). Since $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n$ converges, as noted above, it follows by the Basic Comparison Test that $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{2^n + 5^n}$ converges as well. □

2. *(Limit Comparison Test)* Observe that the dominant terms in the numerator and denominator are 3^n and 5^n, respectively, which suggests that the given series ought to converge or diverge depending on whether $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{5^n}$ does. The latter series does converge, because it is a geometric series with common ratio $r = \frac{3}{5} < 1$.

We have that $\displaystyle\lim_{n\to\infty} \frac{\frac{3^n}{2^n+5^n}}{\frac{3^n}{5^n}} = \lim_{n\to\infty} \frac{3^n}{2^n + 5^n} \cdot \frac{5^n}{3^n} = \lim_{n\to\infty} \frac{5^n}{2^n + 5^n} = \lim_{n\to\infty} \frac{5^n}{2^n + 5^n} \cdot \frac{\frac{1}{5^n}}{\frac{1}{5^n}} = \lim_{n\to\infty} \frac{1}{\frac{2^n}{5^n} + 1} = \frac{1}{0+1} = 1$ – note that $\frac{2^n}{5^n} = \left(\frac{2}{5}\right)^n \to 0$ as $n \to \infty$ because $0 < \frac{2}{5} < 1$. Since $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{5^n}$ converges, as noted above, it follows by the Limit Comparison Test that $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{2^n + 5^n}$ converges as well. ■

Quiz #9. Friday, 29 March. *[10 minutes]*

1. Determine for which values of x the series $\displaystyle\sum_{n=0}^{\infty} n3^n x^n$ converges. *[5]*

SOLUTION. As usual for such problems, we first try the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)3^{n+1}x^{n+1}}{n3^n x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)3x}{n}\right| = 3|x|\cdot \lim_{n\to infty}\frac{n+1}{n}$$
$$= 3|x|\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 3(x|\cdot(1+0) = 3|x|$$

It follows by the Ratio Test that the given series converges when $3|x| < 1$, *i.e.* when $-\frac{1}{3} < x < \frac{1}{3}$, and diverges when $3|x| > 1$, *i.e.* when $x < -\frac{1}{3}$ or $x > \frac{1}{3}$. When $3|x| = 1$, *i.e.* when $x = \pm\frac{1}{3}$, the Ratio Test tells us nothing, so we have to handle these cases in other ways.

If $x = +\frac{1}{3}$, the series becomes $\displaystyle\sum_{n=0}^{\infty} n3^n\left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} n$ which diverges by the Divergence Test because $\displaystyle\lim_{n\to\infty} n = \infty$.

If $x = -\frac{1}{3}$, the series becomes $\displaystyle\sum_{n=0}^{\infty} n3^n\left(-\frac{1}{3}\right)^n = \sum_{n=0}^{\infty}(-1)^n n$ which also diverges by the Divergence Test because $\displaystyle\lim_{n\to\infty}(-1)^n n$ does not exist. (The even terms head off to ∞ and the odd terms head off to $-\infty$.)

Combining the above, the given series converges for, and only for, $-\frac{1}{3} < x < \frac{1}{3}$. ■

Quiz #10. Friday, 5 April. *[15 minutes]*

1. Find the Taylor series about $a = 0$ of $f(x) = \dfrac{1}{(x+1)^2}$. *[3]*
2. Find the radius and interval of convergence of this Taylor series. *[2]*

SOLUTIONS. 1. Recall that the Taylor series of $f(x)$ at a is the power series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$. In our case, we have $a = 0$, so we have to work out what $f^{(n)}(0)$ is for all $n \geq 0$ when $f(x) = \dfrac{1}{(x+1)^2} = (x+1)^{-2}$. We try brute force and pattern recognition:

n	0	1	2	3	$\cdots$
$f^{(n)}(x)$	$(x+1)^{-2}$	$-2(x+1)^{-3}$	$6(x+1)^{-4}$	$-24(x+1)^{-5}$	$\cdots$
$f^{(n)}(0)$	1	-2	6	-24	$\cdots$

It's not too hard to see that in general we have $f^{(n)}(x) = (-1)^n(n+1)!(x+1)^{n+2}$ and therefore $f^{(n)}(0) = (-1)^n(n+1)!1^{n+2} = (-1)^n(n+1)!$. It follows that the taylor series at $a = 0$ of $f(x) = \dfrac{1}{(x+1)^2} = (x+1)^{-2}$ is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)!}{n!}x^n = \sum_{n=0}^{\infty}(-1)^n(n+1)x^n \qquad \square$$

2. As usual, we try the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\left((n+1)+1\right)x^{n+1}}{(-1)^n(n+1)x^n}\right| = \lim_{n\to\infty}\left|\frac{-(n+2)x}{n+1}\right|$$
$$= |x|\lim_{n\to\infty}\frac{n+2}{n+1}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} = |x|\lim_{n\to\infty}\frac{1+\frac{2}{n}}{1+\frac{1}{n}} = |x|\frac{1+0}{1+0} = |x|$$

It follows by the Ratio Test that the Taylor series obtained above converges when $|x| < 1$ and diverges when $|x| > 1$, so its radius of convergence is $R = 1$.

To determine the interval of convergence we also need to check whether the series converges or diverges at its endpoints, $x = \pm 1$:

$x = -1$: In this case the series is $\displaystyle\sum_{n=0}^{\infty}(-1)^n(n+1)(-1)^n = \sum_{n=0}^{\infty}(-1)^{2n}(n+1) = \sum_{n=0}^{\infty}(n+1)$, which diverges by the Divergence Test because $\displaystyle\lim_{n\to\infty}(n+1) = \infty \neq 0$.

$x = +1$: In this case the series is $\displaystyle\sum_{n=0}^{\infty}(-1)^n(n+1)1^n = \sum_{n=0}^{\infty}(-1)^n(n+1)$, which also diverges by the Divergence Test because $\displaystyle\lim_{n\to\infty}(-1)^n(n+1)$ does not exist. (The odd-numbered terms head off to $-\infty$ while the even-numbered terms head off to ∞.)

Putting all this together, it follows that the interval of convergence of this Taylor series is $(-1, 1)$. ■