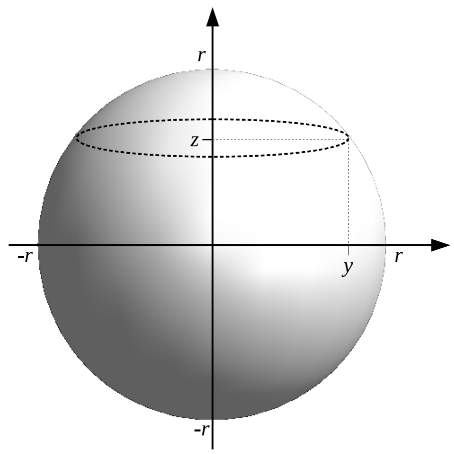
Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2019

Assignment #4
Surface ~~Area~~ Error
Due on Friday, 8 March.

If you look it up, you will find that the surface area of a sphere of radius r is $SA = 4\pi r^2$. The following calculation comes to a different conclusion.

We will compute the surface area of the sphere by adding up (by integrating) the perimeters of horizontal cross-sections of the sphere, analogously to how we can [correctly!] find the volume of a sphere by adding up (by integrating) the areas of horizontal cross sections of the sphere.

The equation of a sphere of radius r centred at the origin is $x^2 + y^2 + z^2 = r^2$. The cross-section of this sphere for a fixed z with $-r \leq z \leq r$ is a circle with equation $x^2 + y^2 = r^2 - z^2$ and hence radius $R(z) = \sqrt{r^2 - z^2}$ and perimeter $C(z) = 2\pi R(z) = 2\pi\sqrt{r^2 - z^2}$. Therefore the surface area of the sphere should be $\int_{-r}^{r} C(z)\,dz = \int_{-r}^{r} 2\pi\sqrt{r^2 - z^2}\,dz$. Let's see what happens when we compute this integral. We will substitute $z = r\sin(\theta)$, so $dz = r\cos(\theta)d\theta$, and change the limits as we go along: $\begin{array}{ccc} z & -r & r \\ \theta & -\pi/2 & \pi/2 \end{array}$.



$$\begin{aligned}
\int_{-r}^{r} C(z)\,dz &= \int_{-r}^{r} 2\pi\sqrt{r^2 - z^2}\,dz = 2\pi \int_{-\pi/2}^{\pi/2} \sqrt{r^2 - r^2\sin^2(\theta)}\, r\cos(\theta)\,d\theta \\
&= 2\pi \int_{-\pi/2}^{\pi/2} \sqrt{r^2\cos^2(\theta)}\, r\cos(\theta)\,d\theta = 2\pi \int_{-\pi/2}^{\pi/2} r^2\cos^2(\theta)\,d\theta \\
&= 2\pi r^2 \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(1 + \cos(2\theta)\right)\,d\theta = \pi r^2 \left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_{-\pi/2}^{\pi/2} \\
&= \pi r^2 \left(\frac{\pi}{2} + \frac{1}{2}\cdot 0\right) - \pi r^2 \left(-\frac{\pi}{2} + \frac{1}{2}\cdot 0\right) = \pi^2 r^2
\end{aligned}$$

This is only true in those universes where $\pi = 4$ ...

1. Explain what is wrong in the calculation above. *[4]*

2. Use calculus to compute the surface area of a sphere of radius r correctly. *[6]*