Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Summer 2018
Solutions to the Final Examination
19:00-22:00 on Monday, 30 July, in CGS 105.

Time: *3 hours.* *Brought to you by* Стефан Біланюк.

Instructions: Do parts **A**, **B**, and **C**, and, if you wish, part **D**. Show all your work and justify all your answers. *If in doubt about something,* **ask!**

Aids: Any calculator; (all sides of) one aid sheet; one (1) brain (no neuron limit).

Part A. Do all four (4) of **1**–**4**.

1. Evaluate any *four* (4) of the integrals **a**–**f**. *[20 = 4 × 5 each]*

$$\textbf{a. } \int_{-1}^{0} \frac{1}{x^2+2x+2}\,dx \qquad \textbf{b. } \int_0^1 \frac{1}{\sqrt{y}}\,dy \qquad \textbf{c. } \int_{-\pi/4}^{\pi/4} \sec^2(z)\tan(z)\,dz$$

$$\textbf{d. } \int \left(1+w^2\right)^{1/2}\,dw \qquad \textbf{e. } \int_0^\infty ve^{-v}\,dv \qquad \textbf{f. } \int \frac{u+1}{u^3-u}\,du$$

Solutions. **a.** Note that $x^2+2x+2 = x^2+2x+1+1 = (x+1)^2+1$. We will use this fact and the substitution $u = x+1$, so $du = dx$ and $\begin{array}{ccc} x & -1 & 0 \\ u & 0 & 1 \end{array}$.

$$\int_{-1}^{0} \frac{1}{x^2+2x+2}\,dx = \int_{-1}^{0} \frac{1}{(x+1)^2+1}\,dx = \int_0^1 \frac{1}{u^2+1}\,du = \arctan(u)\big|_0^1$$
$$= \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4} \quad \square$$

b. Since $f(y) = \dfrac{1}{\sqrt{y}} = y^{-1/2}$ has an asymptote at $y = 0$, $\displaystyle\int_0^1 \frac{1}{\sqrt{y}}\,dy$ is an improper integral.

$$\int_0^1 \frac{1}{\sqrt{y}}\,dy = \lim_{t\to0^+}\int_t^1 y^{-1/2}\,dy = \lim_{t\to0^+} \left.\frac{y^{1/2}}{1/2}\right|_t^1 = \lim_{t\to0^+} 2\sqrt{y}\big|_t^1$$
$$= \lim_{t\to0^+}\left[2\sqrt{1} - 2\sqrt{t}\right] = 2 - 2\sqrt{0} = 2 - 0 = 2 \quad \square$$

c. We'll use the substitution $w = \sec(z)$, so $dw = \sec(z)\tan(z)\,dz$ and $\begin{array}{ccc} z & -\pi/4 & \pi/4 \\ w & \sqrt(2) & \sqrt{2} \end{array}$. Note that $\cos(-\pi/4) = \cos(\pi/4) = \frac{1}{\sqrt{2}}$, so $\sec(-\pi/4)) = \sec(\pi/4)$. Note also that $\int_a^a f(z)\,dz = 0$ for any constant a and integrable function $f(z)$. It follows that:

$$\int_{-\pi/4}^{\pi/4} \sec^2(z)\tan(z)\,dz = \int_{\sqrt{2}}^{\sqrt{2}} w\,dw = 0 \quad \square$$

d. We will use the trigonometric substitution $w = \tan(\theta)$, so $dw = \sec^2(\theta)\,d\theta$, as well as the reduction formula $\displaystyle\int \sec^n(\theta)\,d\theta = \frac{1}{n-1}\sec^{n-2}(\theta)\tan(\theta) + \frac{n-2}{n-1}\int \sec^{n-2}(\theta)\,d\theta$.

$$\begin{aligned}
\int \left(1+w^2\right)^{1/2}\,dw &= \int \left(1+\tan^2(\theta)\right)^{1/2}\sec^2(\theta)\,d\theta = \int \left(\sec^2(\theta)\right)^{1/2}\sec^2(\theta)\,d\theta \\
&= \int \sec(\theta)\sec^2(\theta)\,d\theta = \int \sec^3(\theta)\,d\theta \\
&= \frac{1}{3-1}\sec^{3-2}(\theta)\tan(\theta) + \frac{3-2}{3-1}\int \sec^{3-2}(\theta)\,d\theta \\
&= \frac{1}{2}\tan(\theta)\sec(\theta) + \frac{1}{2}\int \sec(\theta)\,d\theta \\
&= \frac{1}{2}\tan(\theta)\sec(\theta) + \frac{1}{2}\ln\left(\tan(\theta)+\sec(\theta)\right) + C \\
&= \frac{1}{2}w\left(1-w^2\right)^{1/2} + \frac{1}{2}\ln\left(w + \left(1-w^2\right)^{1/2}\right) + C \quad \square
\end{aligned}$$

e. This is obviously an improper intehral, so we will need to compute a limit. To obtain the necessary antiderivative, we will use integration by parts, with $u = v$ and $w' = e^{-v}$, so $u' = 1$ and $w = (-1)e^{-v}$. We will also make use of the fact that $e^{-t} \to 0$ much faster than $t \to \infty$, so that $\displaystyle\lim_{t\to\infty} te^{-t} = 0$.

$$\begin{aligned}
\int_0^\infty ve^{-v}\,dv &= \lim_{t\to\infty}\int_0^t ve^{-v}\,dv = \lim_{t\to\infty}\left[v\cdot(-1)e^{-v}\Big|_0^t - \int_0^t 1\cdot(-1)e^{-v}\,dv\right] \\
&= \lim_{t\to\infty}\left[\left(-te^{-t}\right) - \left(-0e^{-0}\right) + \int_0^t e^{-v}\,dv\right] \\
&= \lim_{t\to\infty}\left[-te^{-t} + 0 + (-1)e^{-v}\Big|_0^t\right] = \lim_{t\to\infty}\left[-te^{-t} + (-1)e^{-t} - (-1)e^{-0}\right] \\
&= \lim_{t\to\infty}\left[-te^{-t} - e^{-t} + 1\right] = -0 - 0 + 1 = 1 \quad \square
\end{aligned}$$

f. Notice that because $u^3 - u = u\left(u^2-1\right) = u(u-1)(u+1)$. , we have:

$$\int \frac{u+1}{u^3-u}\,du = \int \frac{u+1}{u(u-1)(u+1)}\,du = \int \frac{1}{u(u-1)}\,du$$

We will now apply the method of partial fractions.

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1} = \frac{A(u-1)+Bu}{u(u-1)} = \frac{(A+B)u - A}{u(u-1)},$$

so we must have $A + B = 0$ and $-A = 1$, from which it follows that $A = -1$ and $B = -A = -(-1) = 1$. Thus:

$$\begin{aligned}
\int \frac{u+1}{u^3-u}\,du &= \int \frac{1}{u(u-1)}\,du = \int \left(\frac{-1}{u} + \frac{1}{u-1}\right)du = -\int \frac{1}{u}\,du + \int \frac{1}{u-1}\,du \\
&= -\ln(u) + \ln(u-1) + C = \ln\left(\frac{u-1}{u}\right) + C = \ln\left(1 - \frac{1}{u}\right) + C \quad \blacksquare
\end{aligned}$$

2. Determine whether the series converges in any *four* (4) of **a**–**f**. *[20 = 4 × 5 each]*

a. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n}$ **b.** $\displaystyle\sum_{m=1}^{\infty} \frac{(-1)^m}{\sqrt{m!}}$ **c.** $\displaystyle\sum_{\ell=2}^{\infty} \frac{\ell+2}{\ell^{5/2}+\ell^{3/2}+\ell^{1/2}}$

d. $\displaystyle\sum_{k=3}^{\infty} \frac{3}{k\,[\ln(k)]^2}$ **e.** $\displaystyle\sum_{j=4}^{\infty} \frac{j\cos(j)}{(2j)!}$ **f.** $\displaystyle\sum_{i=5}^{\infty} e^{-i}\arctan(i)$

SOLUTIONS. **a.** *(Divergence Test)* Note that $a_7 = \dfrac{7!}{2^7} = \dfrac{5040}{2187} > 1$. Since $\dfrac{n+1}{3} > \dfrac{8}{3} > 1$ when $n \geq 7$, we have $a_{n+1} = \dfrac{(n+1)!}{3^{n+1}} = \dfrac{n!}{3^n} \cdot \dfrac{n+1}{3} > \dfrac{n!}{3^n} = a_n > 1$ for all $n \geq 7$. It follows that $\displaystyle\lim_{n\to\infty} a_n \neq 0$, if the limit exists. Hence the given series diverges by the Divergence Test. □

a. *(Ratio Test)* Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{3^{n+1}}}{\frac{n!}{3^n}}\right| = \lim_{n\to\infty}\frac{(n+1)!}{3^{n+1}}\cdot\frac{3^n}{n!} = \lim_{n\to\infty}\frac{n+1}{3} = \infty > 1\,,$$

it follows that the given series diverges by the Ratio Test. □

b. *(Alternating Series Test)* Since the denominator $\sqrt{m!} > 0$ for all $m \geq 1$ and the numerator $(-1)^m$ alternates, the terms of the series alternate in sign. More over, since $m!$ and $\sqrt{x}$ both increase as m increases, we have $\sqrt{(m+1)!} > \sqrt{m!}$ and hence $\left|\dfrac{(-1)^{m+1}}{\sqrt{(m+1)!}}\right| = \dfrac{1}{\sqrt{(m+1)!}}| < \dfrac{1}{\sqrt{m!}} = \left|\dfrac{(-1)^m}{\sqrt{m!}}\right|$ for all $m \geq 1$. Finally, since $m! \to \infty$ as $m \to \infty$ and $\sqrt{x} \to \infty$ as $x \to \infty$, $\sqrt{m!} \to \infty$ as $m \to \infty$, and so $\dfrac{(-1)^m}{\sqrt{m!}} \to 0$ as $m \to \infty$. Since it satisfies the three conditions of the Alternating Series Test, the series converges. □

b. *(Ratio Test)* Since

$$\begin{aligned}\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| &= \lim_{m\to\infty}\left|\frac{\frac{(-1)^{m+1}}{\sqrt{(m+1)!}}}{\frac{(-1)^m}{\sqrt{m!}}}\right| = \lim_{m\to\infty}\left|\frac{(-1)^{m+1}}{\sqrt{(m+1)!}}\cdot\frac{\sqrt{m!}}{(-1)^m}\right| \\ &= \lim_{m\to\infty}\sqrt{\frac{m!}{(m+1)!}} = \lim_{m\to\infty}\sqrt{\frac{1}{m+1}} = \sqrt{0} = 0 < 1\,,\end{aligned}$$

the given series converges by the Ratio Test. □

c. *(Comparison Test)* For all $\ell \geq 2$, $\dfrac{\ell+2}{\ell^{5/2}+\ell^{3/2}+\ell^{1/2}} \leq \dfrac{\ell+\ell}{\ell^{5/2}} = \dfrac{2\ell}{\ell^{5/2}} = \dfrac{2}{\ell^{3/2}}$. Since the series $\displaystyle\sum_{\ell=2}^{\infty}\frac{2}{\ell^{3/2}}$ converges by the p-Test because $p = \frac{3}{2} - 0 = \frac{3}{2} = 1.5 > 1$, it follows that the given series converges by the Comparison Test. □

c. *(Limit Comparison Test)* Since

$$\lim_{\ell\to\infty} \frac{\frac{\ell+2}{\ell^{5/2}+\ell^{3/2}+\ell^{1/2}}}{\frac{1}{\ell^{3/2}}} = \lim_{\ell\to\infty} \frac{\ell+2}{\ell^{5/2}+\ell^{3/2}+\ell^{1/2}} \cdot \frac{\ell^{3/2}}{1} = \lim_{\ell\to\infty} \frac{\ell^{5/2}+2\ell^{3/2}}{\ell^{5/2}+\ell^{3/2}+\ell^{1/2}}$$
$$= \lim_{\ell\to\infty} \frac{\ell^{5/2}+2\ell^{3/2}}{\ell^{5/2}+\ell^{3/2}+\ell^{1/2}} \cdot \frac{\frac{1}{\ell^{5/2}}}{\frac{1}{\ell^{5/2}}} = \lim_{\ell\to\infty} \frac{1+\frac{2}{\ell}}{1+\frac{1}{\ell}+\frac{1}{\ell^2}}$$
$$= \frac{1+0}{1+0+0} = 1\,,$$

and $0 < 1 < \infty$, the Limit Comparison Test tells us that the given series converges or diverges exactly as the series $\displaystyle\sum_{\ell=2}^{\infty} \frac{1}{\ell^{3/2}}$ does. As this series converges by the p-Test because $p = \frac{3}{2} - 0 = \frac{3}{2} > 1$, it follows that the given series converges too. □

Note. In both of the solutions to **c** above, the use of the p-Test could have been replaced by a somewhat more cumbersome use of the Integral Test.

c. *(Generalized p-Test)* The general term $\dfrac{\ell+2}{\ell^{5/2}+\ell^{3/2}+\ell^{1/2}}$ of the series is a rational function with a denominator of degree $\frac{5}{2}$ and a numerator of degree 1. Since $p = \frac{5}{2} - 1 = \frac{3}{2} = 1.5 > 1$, it follows by the Generalized p-Test that the given series converges. □

d. *(Integral Test)* By the Integral Test, the series $\displaystyle\sum_{k=3}^{\infty} \frac{3}{k\,[\ln(k)]^2}$ converges if and only if the improper integral $\displaystyle\int_3^{\infty} \frac{3}{x\,[\ln(x)]^2}\,dx$ converges, *i.e.* evaluates out to a real number. To evaluate this integral, we will use the substitution $u = \ln(x)$, so $du = \frac{1}{x}\,dx$ and $\begin{matrix} x & 3 & \infty \\ u & \ln(3) & \infty \end{matrix}$. Since

$$\int_3^{\infty} \frac{3}{x\,[\ln(x)]^2}\,dx = \int_{\ln(3)}^{\infty} \frac{3}{u^2}\,du = \lim_{t\to\infty} \int_{\ln(3)}^{t} \frac{3}{u^2}\,du = \lim_{t\to\infty} \left. -\frac{3}{u} \right|_{\ln(3)}^{t}$$
$$= \lim_{t\to\infty} \left[\left(-\frac{3}{t}\right) - \left(-\frac{3}{\ln(3)}\right) \right] = -0 + \frac{3}{\ln(3)} = \frac{3}{\ln(3)}\,,$$

it follows that the given series converges. □

e. *(Comparison Test and Ratio Test)* We will show that the given series converges absolutely, and hence converges. First, $\left|\dfrac{j\cos(j)}{(2j)!}\right| \le \dfrac{j}{(2j)!}$ for all $j \ge 4j$, so $\displaystyle\sum_{j=4}^{\infty} \frac{j\cos(j)}{(2j)!}$ converges absolutely by the Comparison Test if the series $\displaystyle\sum_{j=4}^{\infty} \frac{j}{(2j)!}$ converges. Second, to see that this series converges we apply the Ratio Test. Since

$$\lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}\right| = \lim_{j\to\infty}\left|\frac{\frac{j+1}{(2j+2)!}}{\frac{j}{(2j)!}}\right| = \lim_{j\to\infty}\frac{j+1}{(2j+2)!}\cdot\frac{(2j)!}{j} = \lim_{j\to\infty}\frac{j+1}{(2j+2)(2j+1)j}$$
$$= \lim_{j\to\infty}\frac{1}{2j(2j+1))} = \lim_{j\to\infty}\frac{1}{4j^2+2j} = 0 < 1\,,$$

the series $\displaystyle\sum_{j=4}^{\infty}\frac{j}{(2j)!}$ does converge by the Ratio Test, and so the given series converges absolutely. □

f. *(Comparison Test)* Since $0 < e^{-i}\arctan(i) < \dfrac{\pi}{2}e^{-i} = \dfrac{\pi}{2}\cdot\dfrac{1}{e^i}$ for all $i \geq 5$, the given series $\displaystyle\sum_{i=5}^{\infty} e^{-i}\arctan(i)$ converges by comparison with the series $\displaystyle\frac{\pi}{2}\sum_{i=5}^{\infty}\frac{1}{e^i}$, which converges because it is a geometric series with $|r| = \dfrac{1}{e} < 1$. □

f. *(Ratio Test)* Since

$$\lim_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| = \lim_{i\to\infty}\left|\frac{e^{-i-1}\arctan(i+1)}{e^{-i}\arctan(i)}\right| = \lim_{i\to\infty} e^{-1}\cdot\frac{\arctan(i+1)}{\arctan(i)} = e^{-1}\cdot\frac{\pi/2}{\pi/2} = \frac{1}{e} < 1\,,$$

the Ratio Test implies that the given series converges. ■

3. Do any *four* (4) of **a**–**f**. *[20 = 4 × 5 each]*

a. Find the Taylor series at $a = 0$ of $f(x) = \dfrac{1}{x+1}$.

b. Find the arc-length of the curve $y = \ln(\cos(x))$, where $0 \leq x \leq \dfrac{\pi}{4}$.

c. Suppose $a_0 = a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$. Compute $\lim_{n\to\infty} \dfrac{1}{a_n}$.

d. Find the area of the region between $y = 1$ and $y = e^{-x}$ for $0 \leq x \leq \ln(2)$.

e. Determine the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n+1}x^n}{4^n+1}$.

f. Use the Right-Hand Rule or the Trapezoid Rule to approximate the definite integral $\displaystyle\int_0^1 \sin(\pi x)\,dx$ to within 1 of the exact value.

Solutions. **a.** *(Taylor's Formula)* We compute and evaluate derivatives of $f(x) = \dfrac{1}{x+1}$, looking for patterns:

n	0	1	2	3	$\cdots$	n	$\cdots$
$f^{(n)}(x)$	$\frac{1}{x+1}$	$\frac{-1}{(x+1)^2}$	$\frac{2}{(x+1)^3}$	$\frac{-6}{(x+1)^4}$	$\cdots$	$\frac{(-1)^n n!}{(x+1)^{n+1}}$	$\cdots$
$f^{(n)}(0)$	1	-1	2	-6	$\cdots$	$(-1)^n n!$	$\cdots$

We plug what happens at n into Taylor's Formula to obtain the Taylor series at $a = 0$ for $f(x) = \dfrac{1}{x+1}$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!} x^n = \sum_{n=0}^{\infty} (-1)^n x^n \quad \square$$

a. *(Algebra)* Recall the formula for the sum of a geometric series with first term a and common ratio r, namely $a + ar + ar^2 + \cdots = \frac{a}{1-r}$. Applying this formula in reverse, we see that:

$$f(x) = \frac{1}{x+1} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$$

This must be the Taylor series of $f(x)$ at $a = 0$ because whenever a function is actually equal to a power series at a, that power series is the function's Taylor series at a. $\square$

b. We plug $\dfrac{dy}{dx} = \dfrac{d}{dx}\ln(\cos(x)) = \dfrac{1}{\cos(x)} \cdot \dfrac{d}{dx}\cos(x) = \dfrac{1}{\cos(x)} \cdot (-\sin(x)) = -\tan(x)$ into the arc-length formula, with $0 \leq x \leq \dfrac{\pi}{4}$, and chug away:

$$\begin{aligned}
\text{arc-length} &= \int_0^{\pi/4} ds = \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{\pi/4} \sqrt{1 + (-\tan(x))^2}\, dx \\
&= \int_0^{\pi/4} \sqrt{1 + \tan^2(x)}\, dx = \int_0^{\pi/4} \sqrt{\sec^2(x)}\, dx = \int_0^{\pi/4} \sec(x)\, dx \\
&= \ln\left(\sec(x) + \tan(x)\right)\Big|_0^{\pi/4} \\
&= \ln\left(\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right) - \ln\left(\sec(0) + \tan(0)\right) \\
&= \ln\left(\sqrt{2} + 1\right) - \ln(1 + 0) = \ln\left(\sqrt{2} + 1\right) \quad \square
\end{aligned}$$

c. Observe that $a_n \geq n$ for all n: $a_0 = 1 \geq 0$ and $a_1 = 1 \geq$ by definition, $a_2 = a_1 + a_0 = 1 + 1 = 2 \geq 2$, and for $n \geq 3$, if we already know that $a_{n-1} \geq n - 1$ and $a_{n-2} \geq n - 2$, then $a_n = a_{n-1} + a_{n-2} \geq (n-1) + (n-2) = 2n - 3 \geq n$ since $n \geq 3$. It follows that $\lim_{n\to\infty} a_n \geq \lim_{n\to\infty} n = \infty$, and thus that $\lim_{n\to\infty} \dfrac{1}{a_n} = 0$. $\square$

Note. The sequence a_n in **c** is the famous Fibonacci sequence. In fact, $a_n \to \infty$ very quickly: it turns out that $a_n = \dfrac{\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}}{2^{n+1}\sqrt{5}}$ for $n \geq 0$, so the series actually grows exponentially.

d. $e^{-x} = \dfrac{1}{e^x} \leq 1$ when $x \geq 0$, so the area between $y = 1$ and $y = e^{-x}$ for $0 \leq x \leq \ln(2)$, with the help of the substitution $u = -x$, with $dx = (-1)\, du$ and $\begin{array}{ccc} x & 0 & \ln(2) \\ u & 0 & -\ln(2) \end{array}$, is:

$$\begin{aligned}
A &= \int_0^{\ln(2)} \left(1 - e^{-x}\right)\, dx = \int_0^{-\ln(2)} \left(1 - e^{u}\right)(-1)\, du = \int_{-\ln(2)}^{0} \left(1 - e^{u}\right)\, du \\
&= \left(u - e^u\right)\Big|_{-\ln(2)}^{0} = \left(0 - e^0\right) - \left(-\ln(2) - e^{-\ln(2)}\right) = -1 + \ln(2) + \frac{1}{e^{\ln(2)}} \\
&= -1 + \ln(2) + \frac{1}{2} = \ln(2) - \frac{1}{2} \quad \square
\end{aligned}$$

e. We will use the Ratio Test.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{\frac{2^{(n+1)+1}x^{n+1}}{4^{n+1}+1}}{\frac{2^{n+1}x^n}{4^n+1}}\right| = \lim_{n\to\infty} \left|\frac{2^{n+2}x^{n+1}}{4^{n+1}+1} \cdot \frac{4^n+1}{2^{n+1}x^n}\right| = \lim_{n\to\infty} 2|x| \frac{4^n+1}{4^{n+1}+1} \\
&= 2|x| \lim_{n\to\infty} \frac{4^n+1}{4^{n+1}+1} \cdot \frac{\frac{1}{4^{n+1}}}{\frac{1}{4^{n+1}}} = 2|x| \lim_{n\to\infty} \frac{\frac{1}{4} + \frac{1}{4^{n+1}}}{1 + \frac{1}{4^{n+1}}} = 2|x| \cdot \frac{\frac{1}{4} + 0}{1 + 0} \\
&= 2|x| \cdot \frac{1}{4} = \frac{|x|}{2}
\end{aligned}$$

By the Ratio Test, it follows that the series converges when $\frac{|x|}{2} < 1$, *i.e.* when $|x| < 2$, and diverges when $\frac{|x|}{2} > 1$, *i.e.* when $|x| > 2$. Thus the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{2^{n+1}x^n}{4^n+1}$ is $r = 2$. $\square$

f. *(Right-Hand Rule)* $\left|\frac{d}{dx}\sin(\pi x)\right| = |\pi\cos(\pi x)| = \pi\,|\cos(\pi x)| \leq \pi \cdot 1 = \pi$ for all x, so we may take $M = \pi$ as an upper bound for $\left|\frac{d}{dx}\sin(\pi x)\right|$ on $[0.1]$. Plugging this into the formula for the upper bound on the error given by the Right-Hand Rule sum at n for the integral $\int_0^1 \sin(\pi x)\,dx$ gives us $\frac{M(b-a)^2}{n} = \frac{\pi(1-0)^2}{n} = \frac{\pi}{n}$. To make sure we get the error to be at most 1, it then suffices to have $\frac{\pi}{n} \leq 1$, *i.e.* $\pi \leq n$, and the least integer $n \geq \pi$ is $n = 4$. It remains to compute the corresponding Right-Hand Rule sum:

$$\begin{aligned}
\int_0^1 \sin(\pi x)\,dx &\approx \frac{b-a}{n}\sum_{i=1}^{4} f\left(a + i\frac{b-a}{n}\right) = \frac{1-0}{4}\sum_{i=1}^{4}\sin\left(\pi\left[0 + i\frac{1-0}{4}\right]\right) \\
&= \frac{1}{4}\sum_{i=1}^{4}\sin\left(i\frac{\pi}{4}\right) = \frac{1}{4}\left[\sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{2\pi}{4}\right) + \sin\left(\frac{3\pi}{4}\right) + \sin\left(\frac{4\pi}{4}\right)\right] \\
&= \frac{1}{4}\left[\frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} + 0\right] = \frac{1}{4}\left[1 + \frac{2}{\sqrt{2}}\right] = \frac{1+\sqrt{2}}{4} \approx 0.604 \quad \square
\end{aligned}$$

f. *(Trapezoid Rule)* Since $\frac{d}{dx}\sin(\pi x) = \pi\cos(\pi x)$, we have that

$$\left|\frac{d^2}{dx^2}\sin(\pi x)\right| = \left|\frac{d}{dx}\pi\cos(\pi x)\right| = \pi^2\,|-\sin(\pi x)| \leq \pi^2$$

for all x. We may therefore take $M = \pi^2 < 9.9$ as an upper bound for $\left|\frac{d^2}{dx^2}\sin(\pi x)\right|$ on $[0,1]$. Plugging this into the formula for the upper bound on the error given by the Trapezoid Rule sum at n for the integral $\int_0^1 \sin(\pi x)\,dx$ gives us $\frac{M(b-a)^3}{12n^2} = \frac{\pi^2(1-0)^3}{12n^2} = \frac{\pi}{12n^2}$. To make sure we get the error to be at most 1, it then suffices to have $\frac{\pi^2}{12n^2} < 1$, *i.e.* $\frac{\pi^2}{12} < n^2$. As $\pi^2 < 9.9 < 12$, it is enough to have that $\frac{\pi^2}{12} < 1 \leq n^2$, and the least such positive integer is $n = 1$. It remains to compute the corresponding Trapezoid Rule sum:

$$\begin{aligned}
\int_0^1 \sin(\pi x)\,dx &\approx \frac{b-a}{n} \sum_{i=0}^{n-1} \frac{f\left(a + i\frac{b-a}{n}\right) + f\left(a + (i+1)\frac{b-a}{n}\right)}{2} \\
&= \frac{1-0}{1} \sum_{i=0}^{1-1} \frac{\sin\left(\pi\left[0 + i\frac{1-0}{1}\right]\right) + \sin\left(\pi\left[0 + (i+1)\frac{1-0}{1}\right]\right)}{2} \\
&= 1 \cdot \frac{\sin(\pi \cdot 0) + \sin(\pi \cdot 1)}{2} = \frac{0+0}{2} = 0 \quad \blacksquare
\end{aligned}$$

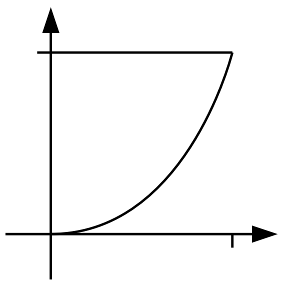
NOTE. We leave it to the interested reader to check that $\displaystyle\int_0^1 \sin(\pi x)dx = \frac{2}{\pi} \approx 0.637$, which is indeed within 1 of both approximations obtained in the solutions to **f** above.

4. Consider the finite region bounded by $x = 0$, $y = 1$, and $y = x^3$.

a. Find the area of this region. *[4]*

b. Find the volume of the solid obtained by revolving the region about $x = 0$. *[8]*

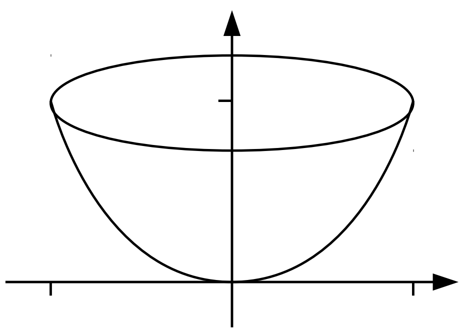
SOLUTIONS. **a.** Here is a sketch of the region:



It is pretty easy to see that the region in question is the one below $y = 1$ and above $y = x^3$ for $0 \le x \le 1$. It follows that the area of the region is:

$$\text{Area} = \int_0^1 \left(1 - x^3\right)\, dx = \left(x - \frac{x^4}{4}\right)\Bigg|_0^1 = \left(1 - \frac{1^4}{4}\right) - \left(0 - \frac{0^4}{4}\right) = \frac{3}{4} - 0 = \frac{3}{4} \quad \square$$

b. Here is a sketch of the solid obtained by revolving the region about $x = 0$, otherwise known as the y-axis:



We will use the method of cylindrical shells. The shell at x has radius $r = x - 0 = x$ and height $h = 1 - x^3$. It follows that the volume of the solid of revolution is:

$$\begin{aligned}\text{Volume} &= \int_0^1 2\pi r h\, dx = \int_0^1 2\pi x \left(1 - x^3\right)\, dx = 2\pi \int_0^1 \left(x - x^4\right)\, dx = 2\pi \left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Bigg|_0^1 \\ &= 2\pi \left(\frac{1^2}{2} - \frac{1^5}{5}\right) - 2\pi \left(\frac{0^2}{2} - \frac{0^5}{5}\right) = 2\pi \cdot \frac{3}{10} - 2\pi \cdot 0 = \frac{3}{5}\pi \quad \blacksquare\end{aligned}$$

NOTE. If one were to instead use the disk method in the solution to **b**, one would have to work in terms of y, and the disk at y would have radius $r = x = y^{1/3}$.

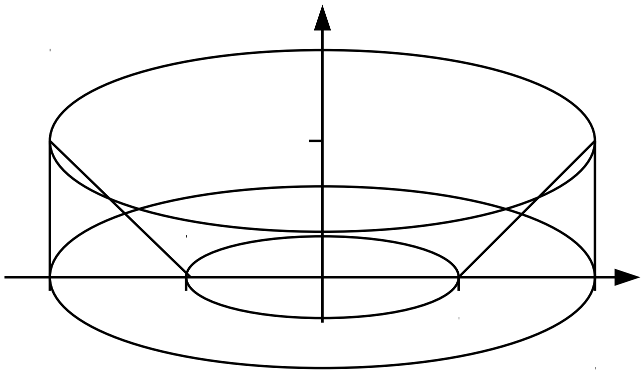
Part B. Do either *one* (1) of **5** or **6**. *[14]*

5. A solid is obtained by revolving the triangle with vertices at $(1, 0)$, $(2, 0)$, and $(2, 1)$ about the y-axis..

a. Find the volume of the solid. *[7]*

b. Find the surface area of the solid. *[7]*

SOLUTIONS. For reference, here is a sketch of the solid of revolution:



Note that the line joining the points $(1, 0)$ and $(2, 1)$ has equation $y = x - 1$, the line joining the points $(1, 0)$ and $(2, 0)$ has equation $y = 0$, and the line joining the points $(2, 0)$ and $(2, 1)$ has equation $x = 2$. Also, note that $1 \leq x \leq 2$ and $0 \leq y \leq 1$ over the triangle.

a. We will use the method of cylindrical shells. The shell at x has radius $r = x - 0 = x$ and height $h = (x - 1) - 0 = x - 1$. It follows that the volume of the solid of revolution is:

$$\text{Volume} = \int_1^2 2\pi r h\, dx = \int_1^2 2\pi x\,(1 - x)\ dx = 2\pi \int_1^2 \left(x - x^2\right)\ dx = 2\pi \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_1^2$$
$$= 2\pi\left(\frac{1^2}{2} - \frac{1^3}{3}\right) - 2\pi\left(\frac{0^2}{2} - \frac{0^3}{3}\right) = 2\pi \cdot \frac{1}{6} - 2\pi \cdot 0 = \frac{\pi}{3} \quad \square$$

NOTE. If one were to instead use the disk/washer method in the solution to **a**, one would have to work in terms of y, and the washer at y would have outer radius $R = 2 - 0 = 2$ and inner radius $r = x = y + 1$.

b. The surface of this solid of revolution has three parts. First, the base, swept out by the base of the triangle when it is revolved, is a washer with outer radius $R = 2 - 0 = 2$ and inner radius $r = 1 - 0 = 1$, and hence area $\pi\left(R^2 - r^2\right) = \pi\left(2^2 - 1^2\right) = 3\pi$. Second, the outside, swept out by the vertical side of the triangle when it is revolved, is a cylinder with radius $r = 2 - 0 = 2$ and height $h = 1 - 0 = 1$, and hence area $2\pi r h = 2\pi \cdot 2 \cdot 1 = 4\pi$. Finally, the remainder is the surface of revolution swept out by the hypotenuse of the triangle, whose area we compute using the appropriate integral formula. Note that the hypotenuse of the triangle is a the piece of the line $y = x - 1$ with $1 \leq x \leq 2$, for which $\dfrac{dy}{dx} = \dfrac{d}{dx}(x - 1) = 1$. Also, note that the point at x on this piece of the line gets revolved through a circle of radius $r = x - 0 = x$. It follows that the remainder has area:

$$\begin{aligned}\text{Area} &= \int_1^2 2\pi r \, ds = 2\pi \int_1^2 x\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = 2\pi \int_1^2 x\sqrt{1+1^2} \, dx = 2\sqrt{2}\pi \int_1^2 x \, dx \\ &= 2\sqrt{2}\pi \left.\frac{x^2}{2}\right|_1^2 = \left.\sqrt{2}\pi x^2\right|_1^2 = \sqrt{2}\pi \cdot 2^2 - \sqrt{2}\pi \cdot 1^2 = 3\sqrt{2}\pi\end{aligned}$$

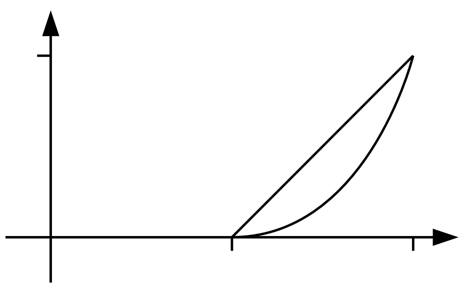
The total surface area of the solid of revolution is thus $3\pi + 4\pi + 3\sqrt{2}\pi = \left(7 + 3\sqrt{2}\right)\pi$. ■

6. Consider the region below $y = x - 1$ and above $y = (x-1)^2$. Find the volume of the solid obtained by revolving this region about ...

a. ... the x-axis. *[7]*

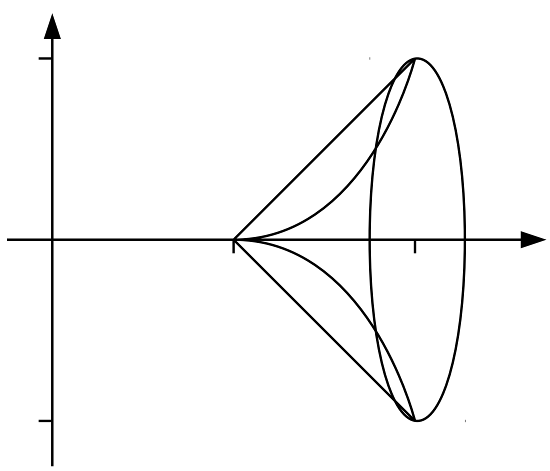
b. ... the y-axis. *[7]*

Solutions. For reference, here is a sketch of the region:



The line $y = x - 1$ and the parabola $y = (x-1)^2$ intersect when $x - 1 = (x-1)^2$, which is only possible when $x - 1 = 0$, *i.e.* when $x = 1$ and $y = 0$, or when $x - 1 = 1$, *i.e.* when $x = 2$ and $y = 1$. Note that $y = x - 1$ is above $y = (x-1)^2$ only for x values between 1 and 2, and that the corresponding y values run from 0 to 1.

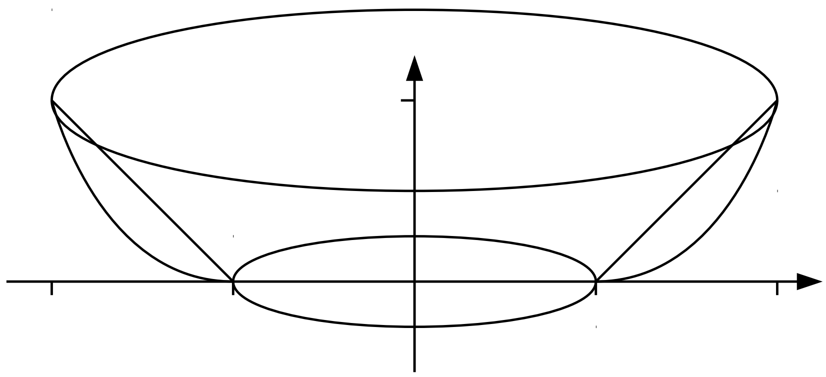
a. Here is a sketch of this solid of revolution:



We will use the disk/washer method. The washer at x for $1 \leq x \leq 2$ has outer radius $R = (x-1) - 0 = x - 1$ and inner radius $r = (x-1)^2 - 0 = (x-1)^2$, and so has area $\pi\left(R^2 - r^2\right) = \pi\left((x-1)^2 - (x-1)^4\right)$. We will use the substitution $u = x - 1$, so $du = dx$ and $\begin{array}{ccc} x & 1 & 2 \\ u & 0 & 1 \end{array}$, to help compute the resulting volume integral.

$$\begin{aligned}
\text{Volume} &= \int_1^2 \pi\left(R^2 - r^2\right)\,dx = \pi\int_1^2 \left((x-1)^2 - (x-1)^4\right)\,dx = \pi\int_0^1 \left(u^2 - u^4\right)\,du \\
&= \pi\left(\frac{u^3}{3} - \frac{u^5}{5}\right)\Big|_0^1 = \pi\left(\frac{1^3}{3} - \frac{1^5}{5}\right) - \pi\left(\frac{0^3}{3} - \frac{0^5}{5}\right) = \pi \cdot \frac{2}{15} - \pi \cdot 0 = \frac{2}{15}\pi \quad \square
\end{aligned}$$

b. Here is a sketch of this solid of revolution:



We will use the method of cylindrical shells. The shell at x for $1 \leq x \leq 2$ has radius $r = x - 0 = x$ and height $h = (x-1) - (x-1)^2 = (x-1) - \left(x^2 - 2x - 1\right) = -x^2 + 3x - 2$, and hence has area $2\pi rh = 2\pi x\left(-x^2 + 3x - 2\right) = 2\pi\left(-x^3 + 3x^2 - 2x\right)$. The volume of this solid of revolution is then:

$$\begin{aligned}
\text{Volume} &= \int_1^2 2\pi rh\,dx = 2\pi\int_1^2 \left(-x^3 + 3x^2 - 2x\right)\,dx = 2\pi\left(-\frac{x^4}{4} + 3\frac{x^3}{3} - 2\frac{x^2}{2}\right)\Big|_1^2 \\
&= 2\pi\left(-\frac{x^4}{4} + x^3 - x^2\right)\Big|_1^2 = 2\pi\left(-\frac{2^4}{4} + 2^3 - 2^2\right) - 2\pi\left(-\frac{1^4}{4} + 1^3 - 1^2\right) \\
&= 2\pi \cdot 0 - 2\pi \cdot \left(-\frac{1}{4}\right) = \frac{\pi}{2} \quad \blacksquare
\end{aligned}$$

Part C. Do either *one* (1) of **7** or **8**. *[14]*

7. Use Taylor's formula to find the Taylor series at $a = 0$ of $f(x) = e^{x+1}$ and determine its radius and interval of convergence.

SOLUTION. We grind out the first few derivatives at $a = 0$ of $f(x) = e^{x+1} = ee^x$ and look for a pattern:

n	0	1	2	3	$\cdots$	n	$\cdots$
$f^{(n)}(x)$	ee^x	ee^x	ee^x	ee^x	$\cdots$	ee^x	$\cdots$
$f^{(n)}(0)$	e	e	e	e	$\cdots$	e	$\cdots$

It should be obvious that $f^{(n)}(0) = e$ for all $n \geq 0$, and so the Taylor series at $a = 0$ of $f(x) = e^{x+1}$ is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{e}{n!} x^n$.

We apply the Ratio Test to discover this series' radius of convergence:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{e}{(n+1)!} x^{n+1}}{\frac{e}{n!} x^n} \right| = \lim_{n\to\infty} \left| \frac{ex^{n+1}}{(n+1)!} \cdot \frac{n!}{ex^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$

for any and all $x \in \mathbb{R}$. It follows that the Taylor series converges for all $x \in \mathbb{R}$, so its radius of convergence is $R = \infty$ and its interval of convergence is $(-\infty . \infty)$. ■

8. Find the Taylor series at $a = 0$ of $f(x) = \dfrac{x}{1 + x^2}$ and determine its radius and interval of convergence.

SOLUTION. We will use a little algebra to find a power series at $a = 0$ which is equal to $f(x) = \dfrac{x}{1 + x^2}$. Recall that the formula for the sum of the infinite geometric series $a + ar + ar^2 + ar^3 + \cdots$ with first term a and common ratio r is $\dfrac{a}{1 - r}$. It is then evindent that $f(x) = \dfrac{x}{1 + x^2} = \dfrac{x}{1 - (-x^2)}$ is the sum of the infinite geometric series with first term $a = x$ and common ratio $r = -x^2$, namely the power series $x - x^3 + x^5 - x^7 + \cdots = \displaystyle\sum_{n=0} (-1)^n x^{2n+1}$.

Since a geometric series converges exactly when $|r| = \left|-x^2\right| = x^2 < 1$ and diverges otherwise, this series converges exactly when $-1 < x < 1$ and diverges otherwise. Thus the radius of convergence of the series is 1 and its interval of convergence is $(-1, 1)$. ■

[Total = 100]

Part D. Bonus problems! If you feel like it and have the time, do one or both of these.

$\bigvee$. The longest straight line that can be drawn entirely on the surface of a perfectly flat and circular road of some constant width is 50 m long. What is the surface area of the road? *[1]*

ANSWER. $625\pi\ m^2$. You figure out why ... :-) ■

$\bigwedge$. Write a haiku (or several :-) touching on calculus or mathematics in general. *[1]*

What is a haiku?

seventeen in three:
five and seven and five of
syllables in lines

SOLUTION. You're on your own! ■

ENJOY THE REST OF YOUR SUMMER!