Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Summer 2018

Assignment #2
The Gamma Function
Due on Monday, 2 July.

One of the big uses of integrals in various parts of mathematics is to define functions that are otherwise difficult to nail down. For example, consider the factorial function on the non-negative integer, defined by $0! = 1$ and $(n+1)! = n! \cdot (n+1)$. (It's pretty easy to check that if $n \geq 1$ is an integer, then $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$.) The factorial function turns up in many parts of mathematics, including algebra, calculus [wait till we do series!], combinatorics, and number theory. The essentially discrete factorial function has a continuous (also differential and integrable) counterpart, which also comes up a fair bit in both applied and theoretical mathematics, namely the gamma function $\Gamma(x)$. This can be defined in a number of different ways, but the easiest definition to work with is in terms of an integral:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt = \lim_{a\to\infty}\int_0^a t^{x-1}e^{-t}\,dt$$

This definition makes sense for all $x > 0$.

1. Verify that $\Gamma(1) = 1$. *[3]*

2. Show that $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$. *[3]*

3. Use **1** and **2** to show that $\Gamma(n+1) = n!$ for every integer $n \geq 0$. *[2]*

4. What is $\Gamma\left(\frac{1}{2}\right)$? *[2]*