Trent University, Summer 2023 (S61)

MATH 1110H Test Solutions

Monday, 29 May

Time: 50 minutes

Name: Nemo Sum

Student Number: 0000000

Question	Mark
1	___
2	___
3	___
Total	___ /30

Instructions

- *Show all your work.* Legibly, please! Simplify where you reasonably can.
- *If you have a question, ask it!*
- Use the back sides of all the pages for rough work or extra space.
- You may use a calculator and (all sides of) one letter- or A4-size aid sheet.
- If you do more than the minimum number of parts, the better ones will count.

1. Do any *two* (2) of parts **a**–**c**. *[10 = 2 × 5 each]*

a. Use the ε–δ definition of limits to verify that $\lim_{x\to -2}(-3x+4) = 10$.

b. Compute $\lim_{x\to 0} \frac{x}{\tan x}$.

c. At what point, if any, does the tangent line to $y = -x^2 + 4x - 3$ have slope 10?

SOLUTION. **a.** Suppose we are given $\varepsilon > 0$. As usual, we will attempt to reverse-engineer the required δ:

$$\begin{aligned} |(-3x+4) - 10| < \varepsilon &\iff |-3x-6| < \varepsilon \iff |(-2)(x+2)| < \varepsilon \\ &\iff 2\,|x+2| < \varepsilon \iff 2\,|x-(-2)| < \varepsilon \\ &\iff |x-(-2)| < \frac{\varepsilon}{2} \end{aligned}$$

Since every step is reversible, if we set $\delta = \frac{\varepsilon}{2}$, then whenever $|x-(-2)| < \delta = \frac{\varepsilon}{2}$, we get $|(-3x+4)-10| < \varepsilon$. Thus, by the ε–δ definition of limits, $\lim_{x\to -2}(-3x+4) = 10$. □

b. *With l'Hôpital's Rule.* Since $x \to 0$ and $\tan(x) \to 0$ as $x \to 0$ we can apply l'Hôpital's Rule to evaluate the limit.

$$\lim_{x\to 0} \frac{x \;\to 0}{\tan x \to 0} = \lim_{x\to 0} \frac{\frac{d}{dx}x}{\frac{d}{dx}\tan x} = \lim_{x\to 0} \frac{1}{\sec^2 x} = \frac{1}{\sec^2(0)} = \frac{1}{1^2} = 1 \quad \Box$$

b. *Without l'Hôpital's Rule.* Recall that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

$$\lim_{x\to 0} \frac{x}{\tan x} = \lim_{x\to 0} \frac{x}{\frac{\sin(x)}{\cos(x)}} = \lim_{x\to 0} \frac{x\cos(x)}{\sin(x)} = \lim_{x\to 0} \frac{\cos(x)}{\frac{\sin(x)}{x}} = \frac{\lim_{x\to 0}\cos(x)}{\lim_{x\to 0}\frac{\sin(x)}{x}} = \frac{\cos(0)}{1} = \frac{1}{1} = 1 \quad \Box$$

c. The slope of the tangent line to the curve $y = -x^2 + 4x - 3$ is given by

$$\frac{dy}{dx} = \frac{d}{dx}\left(-x^2 + 4x - 3\right) = -2x + 4 - 0 = -2x + 4\,.$$

Thus the slope of the tangent line is 10 exactly when

$$-2x + 4 = 10 \iff -2x = 10 - 4 = 6 \iff x = \frac{6}{-2} = -3\,,$$

for which value of x we have $y = -(-3)^2 + 4(-3) - 3 = -9 - 12 - 3 = -24$. Thus the tangent line to the given curve has slope 10 at the point $(-3, -24)$ on the curve. □

2. Find $\dfrac{dy}{dx}$ as best you can in any *two* (2) of parts **a**–**c**. *[10 = 2 × 5 each]*

a. $y = \dfrac{(x-2)^3}{(x-1)^2}$ **b.** $(y-2)^3 = (x-1)^2$ **c.** $y = -\ln\left(\cos\left(x^2\right)\right)$

SOLUTION. **a.** *Quotient Rule, with a side of the Power and Chain Rules.*

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{(x-2)^3}{(x-1)^2}\right) = \frac{\left[\frac{d}{dx}(x-2)^3\right](x-1)^2 - (x-2)^3\left[\frac{d}{dx}(x-1)^2\right]}{\left((x-1)^2\right)^2} \\
&= \frac{\left[3(x-2)^2\frac{d}{dx}(x-2)\right](x-1)^2 - (x-2)^3\left[2(x-1)\frac{d}{dx}(x-1)\right]}{(x-1)^4} \\
&= \frac{3(x-2)^2(x-1)^2 - 2(x-2)^3(x-1)}{(x-1)^4} = \frac{(x-1)(x-2)^2\left[3(x-1)-2(x-2)\right]}{(x-1)^4} \\
&= \frac{(x-2)^2(x+1)}{(x-1)^3} \quad \square
\end{aligned}$$

b. *Implicit differentiation, with a side of the Power and Chain Rules.*

$$\begin{aligned}
(y-2)^3 = (x-1)^2 &\implies \frac{d}{dx}(y-2)^3 = \frac{d}{dx}(x-1)^2 \\
&\implies 3(y-2)^2\frac{d}{dx}(y-2) = 2(x-1)\frac{d}{dx}(x-1) \\
&\implies 3(y-2)^2\frac{dy}{dx} = 2(x-1) \\
&\implies \frac{dy}{dx} = \frac{2(x-1)}{3(y-2)^2} \quad \text{(as long as } y \neq 2\text{)} \quad \square
\end{aligned}$$

b. *Solve for y first, then use the Power and Chain Rules.*

$$\begin{aligned}
(y-2)^3 = (x-1)^2 &\iff y-2 = \left((y-2)^3\right)^{1/3} = \left((x-1)^2\right)^{1/3} = (x-1)^{2/3} \\
&\iff y = (x-1)^{2/3} + 2 \\
&\implies \frac{dy}{dx} = \frac{d}{dx}\left((x-1)^{2/3}+2\right) = \frac{2}{3}(x-1)^{-1/3}\frac{d}{dx}(x-1) + 0 \\
&\implies \frac{dy}{dx} = \frac{2}{3}(x-1)^{-1/3} \quad \square
\end{aligned}$$

c. *Chain Rule, with a side of the Power Rule.*

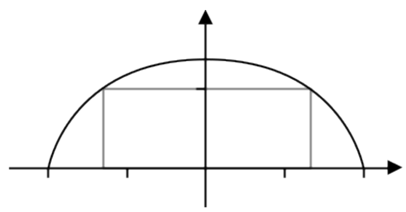
$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(-\ln\left(\cos\left(x^2\right)\right)\right) = -\frac{1}{\cos\left(x^2\right)}\left[\frac{d}{dx}\cos\left(x^2\right)\right] = -\frac{1}{\cos\left(x^2\right)} \cdot \sin\left(x^2\right)\left[\frac{d}{dx}x^2\right] \\
&= -\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)} \cdot 2x = -2x\tan\left(x^2\right) \quad \square
\end{aligned}$$

3. Do *one* (1) of parts **a** or **b**. *[10]*

a. Find the maximum area of a rectangle with each side parallel to one or the other of the x- and y-axes, with two of its corners on the x-axis, and the other two on the part of the parabola $y = \frac{1}{3}\left(4 - x^2\right)$ for which $-2 \leq x \leq 2$.

b. Find all of the vertical and horizontal asymptotes, if any, of $f(x) = \dfrac{x}{\ln(x)}$.

SOLUTION. **a.** Here is a sketch of the setup:



Such a rectangle, with its upper right-hand corner at (x, y) on the parabola for some x with $0 \leq x \leq 2$, has area $A(x) = 2x \cdot y = 2xy = 2x \cdot \dfrac{1}{3}\left(4 - x^2\right) = \dfrac{8}{3}x - \dfrac{2}{3}x^3$. This is what we need to maximize! Note that $A(x)$ is defined, continuous, and differentiable for all relevant x, so we don't, for example, need to worry about vertical asymptotes.

We find the critical points of $A(x)$ in $(0, 2)$.

$$A'(x) = \frac{d}{dx}\left(\frac{8}{3}x - \frac{2}{3}x^3\right) = \frac{8}{3} - \frac{2}{3} \cdot 3x^2 = \frac{8}{3} - 2x^2$$

It follows that $A'(x) = \dfrac{8}{3} - 2x^2 = 0 \iff 2x^2 = \dfrac{8}{3} \iff x = \sqrt{\dfrac{4}{3}} = \dfrac{2}{\sqrt{3}} \approx 1.1547$. Since $0 < \dfrac{2}{\sqrt{3}} < 2$, we have one critical point in the relevant interval. At this point the area is:

$$A\left(\frac{2}{\sqrt{3}}\right) = \frac{8}{3} \cdot \frac{2}{\sqrt{3}} - \frac{2}{3}\left(\frac{2}{\sqrt{3}}\right)^3 = \frac{16}{3\sqrt{3}} - \frac{16}{9\sqrt{3}} = \frac{48 - 16}{9\sqrt{3}} = \frac{32}{9\sqrt{3}} \approx 2.0528$$

The area at the endpoints is $A(0) = \frac{8}{3} \cdot 0 - \frac{2}{3} \cdot 0^3 = 0$ and $A(2) = \frac{8}{3} \cdot 2 - \frac{2}{3} \cdot 2^3 = \frac{16}{3} - \frac{16}{3} = 0$, respectively, so the area at the critical point is the maximum area. $\square$

b. This solution assumes that you are familiar with the natural logarithm function and its graph. If not, please review it first!

$\ln(x)$ is defined, continuous, and differentiable for all $x > 0$ and undefined for $x \leq 0$. It follows that $f(x) = \dfrac{x}{\ln(x)}$ is defined, continuous, and differentiable for all $x > 0$, except when $\ln(x) = 0$, which happens exactly when $x = 1$. Thus we may possibly have a vertical asymptote at $x = 0$, though only on the positive side of 0, and at $x = 1$, on one side or on both. Let's check:

$$\lim_{x \to 0^+} \frac{x \quad \to \quad 0^+}{\ln(x) \to -\infty} = 0^-$$

$$\lim_{x \to 1^-} \frac{x \quad \to 1}{\ln(x) \to 0^-} = -\infty$$

$$\lim_{x \to 1^+} \frac{x \quad \to 1}{\ln(x) \to 0^+} = +\infty$$

Thus we have a vertical asymptote at $x = 1$, with the function reaching for infinity from both directions, but we do not have a vertical asymptote at $x = 0$.

Since $f(x) = \dfrac{x}{\ln(x)}$ is not defined for $x \leq 0$, so the only possible horizontal asymptote would be in the positive direction. Let's check, with a little help from l'Hôpital's Rule:

$$\lim_{x \to +\infty} \frac{x \quad \to +\infty}{\ln(x) \to +\infty} = \lim_{x \to +\infty} \frac{\frac{d}{dx} x}{\frac{d}{dx} \ln(x)} = \lim_{x \to +\infty} \frac{1}{\frac{1}{x}} \lim_{x \to +\infty} x = +\infty$$

This means that the function does not have a horizontal asymptote. $\square$

[Total = 30]