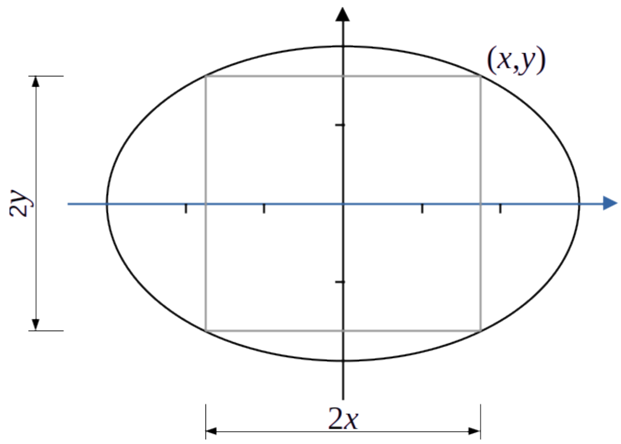
Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2023 (S61)

Quiz #7
Maximum Area

Please give your complete reasoning when answering the question below.

1. Find the maximum area of a rectangle with each side parallel to one or the other of the x- and y-axes and with all of its corners on the ellipse $4x^2 + 9y^2 = 36$. *[10]*

Here is a sketch of the setup:



SOLUTION. Such a rectangle with a corner at (x, y) on the ellipse $4x^2 + 9y^2 = 36$ (or $\frac{x^2}{9} + \frac{y^2}{4} = 1$ in standard form) has area $A(x) = 2x \cdot 2y = 4xy$. We may assume that we're dealing with the corner in the first quadrant, so $0 \le x \le 3$ and $0 \le y \le 2$. Since (x, y) is on the ellipse, we can solve for y in terms of x and plug this into the area function.

$$4x^2 + 9y^2 = 36 \implies 9y^2 = 36 - 4x^2 \implies y^2 = \frac{36 - 4x^2}{9} = \frac{4\left(9 - x^2\right)}{9} = \frac{4}{9}\left(9 - x^2\right)$$
$$\implies y = \sqrt{\frac{4}{9}\left(9 - x^2\right)} = \frac{2}{3}\sqrt{9 - x^2} \quad \text{(Recall that } y \ge 0.\text{)}$$

It follows that we need to maximize $A(x) = 4x \cdot \frac{2}{3}\sqrt{9 - x^2} = \frac{8}{3}x\sqrt{9 - x^2}$ for $0 \le x \le 3$. $A(x)$ is defined and continuous for all $x \in [0, 3]$, so we only need to check for critical points in the interval and compare the values of $A(x)$ at such points with each other and the values of $A(x)$ at the endpoints of the interval.

The endpoints are easy to check: $A(0) = \frac{8}{3} \cdot 0 \cdot \sqrt{9 - 0^2} = 0 \cdot 3 = 0$ and $A(3) = \frac{8}{3} \cdot 3 \cdot \sqrt{9 - 3^2} = 8 \cdot 0 = 0$.

For any critical point(s), we first need to work out where $A'(x)$ is either 0 or undefined for $0 \le x \le 3$.

$$
\begin{aligned}
A'(x) &= \frac{d}{dx}\left(\frac{8}{3}x\sqrt{9-x^2}\right) = \frac{d}{dx}\left(\frac{8}{3}x\left(9-x^2\right)^{1/2}\right) \\
&= \left[\frac{d}{dx}\frac{8}{3}x\right]\left(9-x^2\right)^{1/2} + \frac{8}{3}x\left[\frac{d}{dx}\left(9-x^2\right)^{1/2}\right] \\
&= \frac{8}{3}\left(9-x^2\right)^{1/2} + \frac{8}{3}x \cdot \frac{1}{2}\left(9-x^2\right)^{-1/2}\left[\frac{d}{dx}\left(9-x^2\right)\right] \\
&= \frac{8}{3}\left(9-x^2\right)^{1/2} + \frac{8}{3}x \cdot \frac{1}{2}\left(9-x^2\right)^{-1/2}(-2x) \\
&= \frac{8}{3}\left(9-x^2\right)^{1/2} - \frac{8}{3}x^2\left(9-x^2\right)^{-1/2}
\end{aligned}
$$

Note that $A'(x)$ is undefined at $x = 3$ (Why?), but we have already checked the area at that point, since it is one of the endpoints of the interval. It remains to determine where, if anywhere, $A'(x) = 0$ for $0 < x < 3$.

$$
\begin{aligned}
A'(x) = 0 &\implies \frac{8}{3}\left(9-x^2\right)^{1/2} - \frac{8}{3}x^2\left(9-x^2\right)^{-1/2} = 0 \\
&\implies \left(9-x^2\right)^{1/2} - x^2\left(9-x^2\right)^{-1/2} = 0 \\
&\implies 9 - x^2 - x^2 = 0 \cdot \left(9-x^2\right)^{1/2} = 0 \\
&\implies 9 - 2x^2 = 0 \implies x^2 = \frac{9}{2} \implies x = \pm\sqrt{\frac{9}{2}} = \pm\frac{3}{\sqrt{2}}
\end{aligned}
$$

Since $x = -\dfrac{3}{\sqrt{2}} < 0$, it is not in the interval in question, but $0 < x = \frac{3}{\sqrt{2}} \approx 2.1213 < 3$ is in the interval. The area at this point is:

$$
A\left(\frac{3}{\sqrt{2}}\right) = \frac{8}{3} \cdot \frac{3}{\sqrt{2}} \cdot \sqrt{9 - \left(\frac{3}{\sqrt{2}}\right)^2} = \frac{8}{\sqrt{2}} \cdot \sqrt{9 - \frac{9}{2}} = \frac{8}{\sqrt{2}} \cdot \sqrt{\frac{9}{2}} = \frac{8}{\sqrt{2}} \cdot \frac{3}{\sqrt{2}} = \frac{24}{2} = 12
$$

Since this is the area at the only critical point in the interval, and it exceeds the values of the area at the endpoints of the interval, 12 is the maximum possible area of a rectangle in this set up. □

NOTE. Of course, parts of this would be easier if we used SageMath or a similar program. Solving for y in terms of x, taking the derivative of the area function, solving to find the critical points, and evaluating the area at the relevant critical point, all come to mind. See the next page for these bits done in SageMath.

In [1]:
```
var("y")
solve(4*x^2 + 9*y^2 == 36,y)
```
Out[1]:
```
[y == -2/3*sqrt(-x^2 + 9), y == 2/3*sqrt(-x^2 + 9)]
```
In [2]:
```
A = function('A')(x)
A = (8/3)*x*sqrt(9-(x^2))
diff(A,x)
```
Out[2]:
```
-8/3*x^2/sqrt(-x^2 + 9) + 8/3*sqrt(-x^2 + 9)
```
In [3]:
```
solve(diff(A,x) == 0,x)
```
Out[3]:
```
[x == -3/2*sqrt(2), x == 3/2*sqrt(2)]
```
In [4]:
```
x = 3/2*sqrt(2)
(8/3)*x*sqrt(9-(x^2))
```
Out[4]:
```
12*sqrt(2)*sqrt(1/2)
```
In [5]:
```
N(12*sqrt(2)*sqrt(1/2))
```
Out[5]:
```
12.0000000000000
```