Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2023 (S61)
Solutions to Quiz #5

Do all three of the following questions. Please show all your work and simplify your answers as much as is reasonably possible, which might not be much.

1. Find the derivative of $f(x) = \arccos(x)$ using the fact that $\arccos(x)$ is the inverse function of (part of) $\cos(x)$. *[1.5]*

SOLUTION. Recall from class the general formula for the derivative of an inverse function:

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

We apply this fomula, recollecting also that in this case $f'(x) = \frac{d}{dx}\cos(x) = -\sin(x)$:

$$\frac{d}{dx}\arccos(x) = \frac{1}{-\sin\left(\arccos(x)\right)}$$

It remains to simplify the right-hand side. Since $\sin^2(x) + \cos^2(x) = 1$ for all x, we have that $\sin(x) = \sqrt{1-\cos^2(x)}$. Thus

$$\begin{aligned}\frac{d}{dx}\arccos(x) &= \frac{1}{-\sin\left(\arccos(x)\right)} = \frac{1}{-\sqrt{1-\cos^2\left(\arccos(x)\right)}} \\ &= \frac{-1}{\sqrt{1-\left(\cos\left(\arccos(x)\right)\right)^2}} = \frac{-1}{\sqrt{1-x^2}} \qquad \square\end{aligned}$$

NOTE: One can also do **1** by directly using the method for obtaining the general formula above: start with $\cos\left(\arccos(x)\right) = x$ and differentiate both sides.

2. Compute $\displaystyle\lim_{x\to 1}\frac{e^{(x-1)^2}-1}{(x-1)^3}$. *[1.5]*

SOLUTION. Observe that as $x \to 1$, $(x-1)^2 \to 0$, so $e^{(x-1)^2} - 1 \to e^0 - 1 = 1 - 1 = 0$, and $(x-1)^3 \to 0$ too. The given limit is therefore one we can apply l'Hôpital's Rule to:

$$\begin{aligned}\lim_{x\to 1}\frac{e^{(x-1)^2}-1 \to 0}{(x-1)^3 \quad\to 0} &= \lim_{x\to 1}\frac{\frac{d}{dx}\left(e^{(x-1)^2}-1\right)}{\frac{d}{dx}(x-1)^3} = \lim_{x\to 1}\frac{e^{(x-1)^2}\left[\frac{d}{dx}(x-1)^2\right]-0}{3(x-1)^2\left[\frac{d}{dx}(x-1)\right]} \\ &= \lim_{x\to 1}\frac{e^{(x-1)^2}2(x-1)\left[\frac{d}{dx}(x-1)\right]}{3(x-1)^2(1-0)} = \lim_{x\to 1}\frac{2(x-1)e^{(x-1)^2}(1-0)}{3(x-1)^2} \\ &= \lim_{x\to 1}\frac{2e^{(x-1)^2} \to\ 2e^0 = 2}{3(x-1) \to 3\cdot 0 = 0} \quad \text{so the limit does not exist.}\end{aligned}$$

In fact, it fails to exist in a fairly bad way. If $x \to 1^-$, *i.e.* x approaches 1 from the left, then $x-1$ approaches 0 through negative numbers, so $\displaystyle\lim_{x\to 1^-} \frac{2e^{(x-1)^2} \to 2e^0 = 2}{3(x-1) \to 3\cdot 0^- = 0^-} = -\infty$. On the other hand, if $x \to 1^+$, *i.e.* x approaches 1 from the right, then $x-1$ approaches 0 through positive numbers, so $\displaystyle\lim_{x\to 1^+} \frac{2e^{(x-1)^2} \to 2e^0 = 2}{3(x-1) \to 3\cdot 0^+ = 0^+} = +\infty$. □

NOTE. The curve in question **3** below is the cardioid that appeared in question **2d** of Assignment #1. It also showed up in question **3d** of that assignment with a parametric presentation.

3. Find $\dfrac{dy}{dx}$ at $(x,y) = \left(-1, \sqrt{3+2\sqrt{3}}\right)$ if $\left(x^2+y^2\right)^2 + 4x\left(x^2+y^2\right) - 4y^2 = 0$. *[2]*

SOLUTION. As usual when doing implicit differentiation, we differentiate both sides of the given equation, solve for $\frac{dy}{dx}$, and then plug in the coordinates of the given point.

$$
\begin{aligned}
& \left(x^2+y^2\right)^2 + 4x\left(x^2+y^2\right) - 4y^2 = 0 \\
\implies & \frac{d}{dx}\left(\left(x^2+y^2\right)^2 + 4x\left(x^2+y^2\right) - 4y^2\right) = \frac{d}{dx}0 \\
\implies & 2\left(x^2+y^2\right)\frac{d}{dx}\left(x^2+y^2\right) + \left[\frac{d}{dx}(4x)\right]\left(x^2+y^2\right) + 4x\left[\frac{d}{dx}\left(x^2+y^2\right)\right] \\
& \qquad - \left[\frac{d}{dy}\left(4y^2\right)\right]\frac{dy}{dx} = 0 \\
\implies & 2\left(x^2+y^2\right)\left(2x+2y\frac{dy}{dx}\right) + 4\left(x^2+y^2\right) + 4x\left(2x+2y\frac{dy}{dx}\right) - 8y\frac{dy}{dx} = 0 \\
\implies & 4x\left(x^2+y^2\right) + 4y\left(x^2+y^2\right)\frac{dy}{dx} + 4\left(x^2+y^2\right) + 8x^2 + 8xy\frac{dy}{dx} - 8y\frac{dy}{dx} = 0 \\
\implies & \left[4x\left(x^2+y^2\right) + 4\left(x^2+y^2\right) + 8x^2\right] + \left[4y\left(x^2+y^2\right) + 8xy - 8y\right]\frac{dy}{dx} = 0 \\
\implies & \left[4y\left(x^2+y^2\right) + 8xy - 8y\right]\frac{dy}{dx} = -\left[4x\left(x^2+y^2\right) + 4\left(x^2+y^2\right) + 8x^2\right] \\
\implies & \frac{dy}{dx} = -\frac{4x\left(x^2+y^2\right) + 4\left(x^2+y^2\right) + 8x^2}{4y\left(x^2+y^2\right) + 8xy - 8y} = -\frac{(4x+12)x^2 + (4x+4)y^2}{4y\left(x^2+y^2+2x-2\right)} \\
\implies & \frac{dy}{dx} = -\frac{4(x+3)x^2 + 4(x+1)y^2}{4y\left(x^2+y^2+2x-2\right)} = -\frac{(x+3)x^2 + (x+1)y^2}{y\left(x^2+y^2+2x-2\right)}
\end{aligned}
$$

Now we need to evaluate this expression for $\frac{dy}{dx}$ when $x = -1$ and $y = \sqrt{3+2\sqrt{3}}$.

Here we go:

$$
\begin{aligned}
\left.\frac{dy}{dx}\right|_{(x,y)=\left(-1,\sqrt{3+2\sqrt{3}}\right)} &= -\left.\frac{(x+3)x^2+(x+1)y^2}{y\left(x^2+y^2+2x-2\right)}\right|_{(x,y)=\left(-1,\sqrt{3+2\sqrt{3}}\right)} \\
&= -\frac{(-1+3)(-1)^2+(-1+1)\left(\sqrt{3+2\sqrt{3}}\right)^2}{\sqrt{3+2\sqrt{3}}\left((-1)^2+\left(\sqrt{3+2\sqrt{3}}\right)^2+2(-1)-2\right)} \\
&= -\frac{2+0\cdot\left(3+2\sqrt{3}\right)}{\sqrt{3+2\sqrt{3}}\left(1+3+2\sqrt{3}-2-2\right)} \\
&= -\frac{2}{\sqrt{3+2\sqrt{3}}\cdot 2\sqrt{3}} = \frac{-1}{\sqrt{9+6\sqrt{3}}} \approx -0.2271 \qquad \square
\end{aligned}
$$

NOTE. Before trying to evaluate the expression for $\frac{dy}{dx}$ at the given point, I was just a bit sceptical that I had it right, so I gave the problem to SageMath to see what it got:

```
In [1]: var("y")
        y(x) = function('y')(x)
        cardioid = (x^2 + y^2)^2 + 4*x*(x^2 + y^2) - 4*y^2 == 0
        diff(cardioid,x)

Out[1]: x |--> 4*(x^2 + y(x)^2)*(y(x)*diff(y(x), x) + x) + 8*(y(x)*diff(y(x), x) +
        x)*x + 4*x^2 + 4*y(x)^2 - 8*y(x)*diff(y(x), x) == 0

In [2]: difcard = diff(cardioid,x)
        solve(difcard,diff(y(x),x))

Out[2]: [diff(y(x), x) == -(x^3 + (x + 1)*y(x)^2 + 3*x^2)/(y(x)^3 + (x^2 + 2*x - 2)
        *y(x))]
```

Fortunately, it's the same thing I got, albeit arranged a little differently. One could also use SageMath as a calculator to work out $\frac{dy}{dx}$ when $x=-1$ and $y=\sqrt{3+2\sqrt{3}}$ once one had the geeneral expression. In real life, it might also have been a good idea to use SageMath to check that the point $\left(-1,\sqrt{3+2\sqrt{3}}\right)$ is actually on the cardioid.