Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals

TRENT UNIVERSITY, Summer 2023 (S61)

Solutions to Quiz #1

Do both of the following questions.

1. Use the ε–δ definition of limits to verify that $\lim_{x\to 3}(4-2x) = -2$. *[3]*

SOLUTION. Recall that $\lim_{x\to 3}(4-2x) = -2$ means:

For every $\varepsilon > 0$, there is a $\delta > 0$, such that
if $|x-3| < \delta$, then $|(4-2x)-(-2)| < \varepsilon$.

Suppose that we are given some arbitrary $\varepsilon > 0$. As usual, we will find a corresponding $\delta > 0$ by reverse-engineering: starting with $|(4-2x)-(-2)| < \varepsilon$, we will backwards towards an inequality of the form $|x-3| < \delta$. Here goes:

$$\begin{aligned} |(4-2x)-(-2)| < \varepsilon &\iff |4-2x+2| < \varepsilon \\ &\iff |6-2x| < \varepsilon \\ &\iff |2(3-x)| < \varepsilon \\ &\iff 2\,|3-x| < \varepsilon \\ &\iff |3-x| < \frac{\varepsilon}{2} \\ &\iff |x-3| < \frac{\varepsilon}{2} \quad \text{since } |3-x| = |x-3| \end{aligned}$$

Note that every step is reversible. It follows that if we let $\delta = \frac{\varepsilon}{2}$ and have some x with $|x-3| < \delta = \frac{\varepsilon}{2}$, then $|(4-2x)-(-2)| < \varepsilon$, as desired. Observe that this procedure works no matter what $\varepsilon > 0$ we are given.

Thus, by the ε–δ definition of limits, $\lim_{x\to 3}(4-2x) = -2$. □

2. Compute $\lim_{x\to 1} \frac{x^3-1}{x-1}$ using algebra and the limit laws. *[2]*

SOLUTION. Since $f(x) = \frac{(x^2-1)\cdot 2^x}{(x-1)\cdot 3^x}$ is, like all rational functions, continuous wherever it is defined, we could just plug in $x = 1$ and evaluate to compute the limit if only the function was defined at $x = 1$. Unfortunately, because of the $x-1$ in the denominator, the function we're taking the limit of at $x = 1$ is not defined at $x = 1$. (Dividing by 0 is not recommended for your mental health!) We can get around this by observing that $x^3 - 1 = (x-1)\left(x^2+x+1\right)$, which lets us compute the limit using a little cancellation:

$$\begin{aligned} \lim_{x\to 1} \frac{x^3-1}{x-1} &= \lim_{x\to 1} \frac{(x-1)\left(x^2+x+1\right)}{x-1} \\ &= \lim_{x\to 1} \frac{\left(x^2+x+1\right)}{1} \quad \text{cancelling the } (x-1)\text{s} \\ &= \lim_{x\to 1} \left(x^2+x+1\right) \quad \text{which is continuous everywhere, so} \dots \\ &= \left(1^2+1+1\right) = 3 \end{aligned}$$

Note that every polynomial is continuous at every point, so we can compute their limits at a point just by evaluating them at that point. □