Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2023 (S61)

Solutions to Assignment #5
Right-Hand Rule

Recall from class that the Right-Hand Rule for computing the definite integral of $f(x)$, *i.e.* weighted area between $y = f(x)$ and the x-axis, for x between a and b, is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n} \frac{b-a}{n} \cdot f\left(a + i \cdot \frac{b-a}{n}\right)\right]$$

This actually works as a definition of the definite integral when $f(x)$ is nice enough, such as when it is continuous on $[a, b]$, but even then some basic properties of definite integrals are hard to prove. As a computational method for calculating definite integrals, it's not very useful because even simple integrals can take a while to work through. (See the example we did in class.) In this assignment, you will be asked to do so anyway ... :-)

1. Use the Right-Hand Rule to compute $\displaystyle\int_1^4 \left(x^2+1\right)\,dx$ by hand. *[6]*

You may find the summation formulas $\displaystyle\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$ and $\displaystyle\sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$ to be useful in working through **1**.

SOLUTION. We have $a = 1$, $b = 4$, and $f(x) = x^2 + 1$. We plug these into the Right-Hand Rule formula and work away:

$$\begin{aligned}
\int_1^4 \left(x^2+1\right)\,dx &= \lim_{n\to\infty}\left[\sum_{i=1}^{n} \frac{4-1}{n} \cdot \left(\left(1 + i \cdot \frac{4-1}{n}\right)^2 + 1\right)\right] \\
&= \lim_{n\to\infty}\left[\sum_{i=1}^{n} \frac{3}{n} \cdot \left(\left(1 + i \cdot \frac{3}{n}\right)^2 + 1\right)\right] \\
&= \lim_{n\to\infty}\left[\frac{3}{n} \cdot \sum_{i=1}^{n} \left(1 + 2i \cdot \frac{3}{n} + \left(i \cdot \frac{3}{n}\right)^2 + 1\right)\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \cdot \left[\sum_{i=1}^{n} \left(2 + \frac{6i}{n} + \frac{9i^2}{n^2}\right)\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \cdot \left[\left(\sum_{i=1}^{n} 2\right) + \left(\sum_{i=1}^{n} \frac{6i}{n}\right) + \left(\sum_{i=1}^{n} \frac{9i^2}{n^2}\right)\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \cdot \left[2\left(\sum_{i=1}^{n} 1\right) + \frac{6}{n}\left(\sum_{i=1}^{n} i\right) + \frac{9}{n^2}\left(\sum_{i=1}^{n} i^2\right)\right]
\end{aligned}$$

At this point we replace each of the sums with the appropriate summation formula, per the comment after the question. Continuing:

$$\begin{aligned}
\int_1^4 \left(x^2+1\right)\,dx &= \lim_{n\to\infty} \frac{3}{n} \cdot \left[2\left(\sum_{i=1}^{n} 1\right) + \frac{6}{n}\left(\sum_{i=1}^{n} i\right) + \frac{9}{n^2}\left(\sum_{i=1}^{n} i^2\right)\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \cdot \left[2\cdot n + \frac{6}{n}\cdot\frac{n(n+1)}{2} + \frac{9}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \cdot \left[2n + 3(n+1) + \frac{3}{2}\cdot\frac{(n+1)(2n+1)}{n}\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \cdot \left[5n + 3 + \frac{3}{2}\cdot\frac{2n^2+3n+1}{n}\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \cdot \left[5n + 3 + 3n + \frac{9}{2} + \frac{3}{2n}\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \cdot \left[8n + \frac{15}{2} + \frac{3}{2n}\right] \\
&= \lim_{n\to\infty} \left[\frac{3}{n}\cdot 8n + \frac{3}{n}\cdot\frac{15}{2} + \frac{3}{n}\cdot\frac{3}{2n}\right] \\
&= \lim_{n\to\infty} \left[24 + \frac{45}{2n} + \frac{9}{2n^2}\right] = 24+0+0 = 24 \qquad \square
\end{aligned}$$

2. Use the Right-Hand Rule to compute $\int_1^4 \left(x^2+1\right)\,dx$ using SageMath. *[4]*

You may find SageMath's `sum` command, introduced in the lab of 2023-05-31, and `limit` command, introduced in the lab of 2023-05-10, to be of use in working through **2.**

SOLUTION. Here we go.

```
In [1]: var("n")
        var("i")
        f = function('f')(x)
        f(x) = x^2 + 1
        a = 1
        b = 4
        s = function('s')(n)
        s(n) = sum( (b-a)/n * f(a+i*(b-a)/n),i, 1, n )
        limit(s(n),n=oo)

Out[1]: 24
```

```
In [2]: integral(f,x,a,b)

Out[2]: 24
```

Note that this bit of code is fairly general, so one can change the interval of integration and the integrand (*i.e.* the function being integrated) conveniently. It should work pretty smoothly as long as $f(x)$ is a low-degree polynomial. We also have a check to see if the answer is correct using SageMath's `integral` command. $\square$