Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Summer 2021 (S62)

Solutions to Assignment #2
The Beta Function

One of the ways in which integration is used in mathematics is to define functions. One example of this is defining the natural logarithm function by $\ln(x) = \int_1^x \frac{1}{t}\,dt$ for $x > 0$, instead of as the inverse function to e^x. We will look at another example in this assignment, defining the Beta function $\mathbf{B}(x,y)$ for $x > 0$ and $y > 0$ by:

$$\mathbf{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

Note that as far as evaluating this definite integral is concerned, x and y are treated as constants, since x and y are unaffected by t. The Beta function is pretty useful in various parts of mathematics: the Beta distribution in statistics is defined using this function, it arises in the solutions to various systems of differential equations, has connections with the binomial coefficients, and so on. It can be defined in other ways, but the definition given above is the standard one because it is probably the simplest.

Your task in this assignment will be use what you have learned about integration so far, applied to the definition above, to develop some of the basic facts about the Beta function. In what follows, you may assume that $x > 0$ and $y > 0$ are unknown fixed (*i.e.* constant) real numbers.

1. Verify that $\mathbf{B}(x,y) = \mathbf{B}(y,x)$. *[2]*

Solution. We will use the substitution $u = 1-t$, so $du = (-1)\,dt$ and $dt = (-1)\,du$, and change the limits as we go along, $\begin{matrix} x & 0 & 1 \\ u & 1 & 0 \end{matrix}$. Note that $u = 1-t$ implies that $t = 1-u$. Then:

$$\begin{aligned}\mathbf{B}(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \int_1^0 (1-u)^{x-1}u^{y-1}\,(-1)\,du \\ &= -\int_1^0 u^{y-1}(1-u)^{x-1}\,du = \int_0^1 u^{y-1}(1-u)^{x-1}\,du = \mathbf{B}(y,x) \quad \square\end{aligned}$$

2. Verify that $\mathbf{B}(1,y) = \frac{1}{y}$. *[2]*

Solution. We will use the same substitution that was used in solving **a** above. Note that $t^0 = 1$ for all t and recall that we must have $y > 0$. Then:

$$\begin{aligned}\mathbf{B}(1,y) &= \int_0^1 t^{1-1}(1-t)^{y-1}\,dt = \int_0^1 t^0(1-t)^{y-1}\,dt \int_0^1 (1-t)^{y-1}\,dt = \int_1^0 u^{y-1}\,(-1)\,du \\ &= -\int_1^0 u^{y-1}\,du = \int_0^1 u^{y-1}\,du = \left.\frac{u^y}{y}\right|_0^1 = \frac{1^y}{y} - \frac{0^y}{y} = \frac{1}{y} - 0 = \frac{1}{y} \quad \square\end{aligned}$$

3. Verify that $\mathbf{B}(x, y+1) + \mathbf{B}(x+1, y) = \mathbf{B}x, y)$. *[2]*

Solution. Nothing to see but some very basic properties of the definite integral and a bit of algebra:

$$\begin{aligned}\mathbf{B}(x, y+1) + \mathbf{B}(x+1, y) &= \int_0^1 t^{x-1}(1-t)^{y+1-1}\,dt + \int_0^1 t^{x+1-1}(1-t)^{y-1}\,dt \\ &= \int_0^1 t^{x-1}(1-t)^{y}\,dt + \int_0^1 t^{x}(1-t)^{y-1}\,dt \\ &= \int_0^1 \left[t^{x-1}(1-t)^{y} + t^{x}(1-t)^{y-1}\right]\,dt \\ &= \int_0^1 t^{x-1}(1-t)^{y-1}\left[(1-t)+t\right]\,dt \\ &= \int_0^1 t^{x-1}(1-t)^{y-1}\cdot 1\,dt \\ &= \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \mathbf{B}x, y) \quad \square\end{aligned}$$

4. Verify that $\mathbf{B}(x+1, y) = \dfrac{x}{y}\cdot\mathbf{B}(x, y+1)$. *[3]*

Solution. We will use integration by parts with $u = t^x$ and $v' = (1-t)^{y-1}$, from which it is pretty easy to get $u' = xt^{x-1}$ and $v = -\frac{1}{y}(1-t)^y$. [I'll let you verify that v is correct.]

$$\begin{aligned}\mathbf{B}(x+1, y) &= \int_0^1 t^{x+1-1}(1-t)^{y-1}\,dt = \int_0^1 t^{x}(1-t)^{y-1}\,dt \\ &= t^x\cdot\frac{-1}{y}(1-t)^y\Big|_0^1 - \int_0^1 xt^{x-1}\cdot\frac{-1}{y}(1-t)^y\,dt \\ &= \left(1^x\cdot\frac{-1}{y}0^y - 0^x\cdot\frac{-1}{y}1^y\right) + \frac{x}{y}\int_0^1 t^{x-1}(1-t)^{y-1}\,dt \\ &= (0-0) + \frac{x}{y}\cdot\mathbf{B}(x, y) \quad \square\end{aligned}$$

5. Use the equations in **3** and **4** to verify that $\mathbf{B}(x+1, y) = \dfrac{x+y}{x}\cdot\mathbf{B}(x, y)$. *[1]*

Solution. Using the equations in **3** and **4**, as well as **1**, we have:

$$\begin{aligned}\mathbf{B}(x, y) &= \mathbf{B}(x, y+1) + \mathbf{B}(x+1, y) = \mathbf{B}(y+1, x) + \mathbf{B}(x+1, y) \\ &= \frac{y}{x}\cdot\mathbf{B}(y, x+1) + \frac{x}{x}\cdot\mathbf{B}(x+1, y) = \frac{y}{x}\cdot\mathbf{B}(x+1, y) + \frac{x}{x}\cdot\mathbf{B}(x+1, y) \\ &= \frac{x+y}{x}\cdot\mathbf{B}(x+1, y) \implies \text{So the given equation was a bit off \ldots :-)} \quad \blacksquare\end{aligned}$$

Note. The sharp-eyed will have noticed that when $x < 0$ and/or $y < 0$, the defining integral of $\mathbf{B}(x, y)$ is improper. Feel free to rewrite the above calculations with the appropriate limits in those cases.