Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Summer 2021 (S62)
Solutions to Quiz #3 – *corrected*
Wednesday, 7 July.

Do all of the following questions. Show all your reasoning in each solution. Please note that part marks are available in questions worth more than 0.5 points, so incomplete or incorrect solutions may still earn something.

1. Find the area of the region enclosed by the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$. (Without just using the formula for the area of an ellipse.) *[1]*

Solution. Solving the given equation for y,

$$\frac{x^2}{25} + \frac{y^2}{4} = 1 \implies \frac{y^2}{4} = 1 - \frac{x^2}{25} \implies \frac{y}{2} = \pm\sqrt{1 - \frac{x^2}{25}} \implies y = \pm 2\sqrt{1 - \frac{x^2}{25}}\,,$$

we can see that the region lies below $y = +2\sqrt{1 - \dfrac{x^2}{25}}$ and above $y = -2\sqrt{1 - \dfrac{x^2}{25}}$, both of which expressions make sense as long as $1 - \dfrac{x^2}{25} \geq 0$, *i.e.* when $-5 \leq x \leq 5$. It follows that the area of the region is given by:

$$A = \int_{-5}^{5} \left(2\sqrt{1 - \frac{x^2}{25}} - \left(-2\sqrt{1 - \frac{x^2}{25}} \right) \right) dx = 4 \int_{-5}^{5} \sqrt{1 - \frac{x^2}{25}}\, dx$$

We will solve this integral using the trigonometric substitution $x = 5\sin(\theta)$, so $dx = 5\cos(\theta)\, d\theta$, keeping the old limits and substituting back to x after integrating before using them. Note that $\sin(\theta) = \dfrac{x}{5}$, $\cos(\theta) = \sqrt{1 - \sin^2(\theta)} = \sqrt{1 - \dfrac{x^2}{25}}$, and $\theta = \arcsin\left(\dfrac{x}{5}\right)$.

$$\begin{aligned}
A &= 4 \int_{-5}^{5} \sqrt{1 - \frac{x^2}{25}}\, dx = 4 \int_{x=-5}^{x=5} \sqrt{1 - \frac{25\sin^2(\theta)}{25}}\; 5\cos(\theta)\, d\theta \\
&= 20 \int_{x=-5}^{x=5} \sqrt{1 - \sin^2(\theta)}\, \cos(\theta)\, d\theta = 20 \int_{x=-5}^{x=5} \cos^2(\theta)\, d\theta \\
&= 20 \left[\frac{1}{2}\cos(\theta)\sin(\theta) + \frac{2-1}{2} \int \cos^{2-2}(\theta)\, d\theta \right]_{x=-5}^{x=5} \\
&= 10 \left[\sin(\theta)\cos(\theta) + \int 1\, d\theta \right]_{x=-5}^{x=5} = 10 \left[\sin(\theta)\cos(\theta) + \theta \right]_{x=-5}^{x=5} \\
&= 10 \left[\frac{x}{5}\sqrt{1 - \frac{x^2}{25}} + \arcsin\left(\frac{x}{5}\right) \right]_{x=-5}^{x=5} \\
&= 10 \left[\frac{5}{5}\sqrt{1 - \frac{5^2}{25}} + \arcsin\left(\frac{5}{5}\right) \right] - 10 \left[\frac{-5}{5}\sqrt{1 - \frac{(-5)^2}{25}} + \arcsin\left(\frac{-5}{5}\right) \right] \\
&= 10\left[1 \cdot 0 + \arcsin(1)\right] - 10\left[(-1)\cdot 0 + \arcsin(-1)\right] = 10\left[0 + \frac{\pi}{2}\right] - 10\left[0 - \frac{\pi}{2}\right] \\
&= 5\pi + 5\pi = 10\pi
\end{aligned}$$

Looking up the formula for the area of the ellipse, we get that the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab. In our case, $a = 5$ and $b = 2$, so the enclosed region has area $\pi \cdot 5 \cdot 2 = 10\pi$, so there is some hope our calculation above was correct. :-) ■

2. Find the area of the region below $\dfrac{x}{1+x^4}$ and above $y = 0$, where $0 \le x < \infty$. *[2]*

SOLUTION. Note that $\dfrac{x}{1+x^4} \ge 0$ for all $x \ge 0$, so there aren't any complications arising from the curves crossing each other over the given interval. We set up the area integral, which is an improper integral due to the extent of the given interval, and compute away:

$$A = \int_0^\infty \left(\frac{x}{1+x^4} - 0 \right) dx = \lim_{a \to \infty} \int_0^a \frac{x}{1+x^4}\, dx$$

Substitute $u = x^2$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, and change the limits accordingly: $\begin{array}{ccc} x & 0 & a \\ u & 0 & a^2 \end{array}$

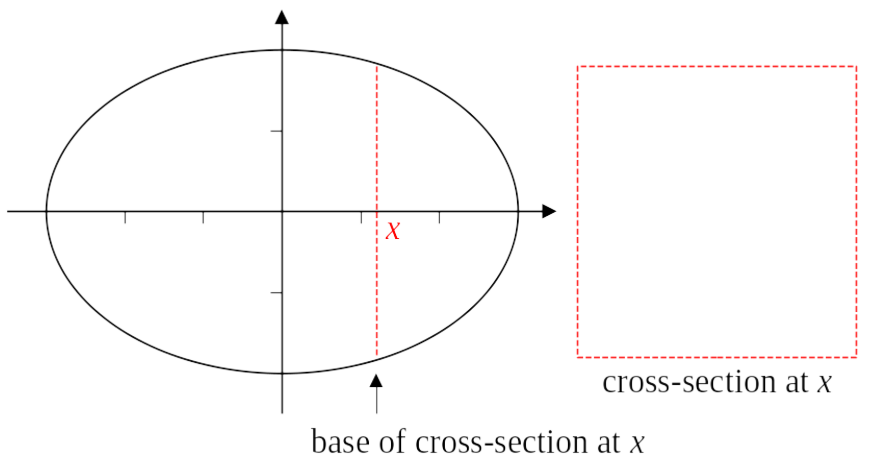
$$= \lim_{a \to \infty} \int_0^{a^2} \frac{1}{1+u^2} \frac{1}{2}\, du = \lim_{a \to \infty} \frac{1}{2} \arctan(u) \Big|_0^{a^2} = \lim_{a \to \infty} \left[\frac{1}{2} \arctan\left(a^2\right) - \frac{1}{2} \arctan(0) \right]$$

$$= \lim_{a \to \infty} \left[\frac{1}{2} \arctan\left(a^2\right) - \frac{1}{2} \cdot 0 \right] = \lim_{a \to \infty} \frac{1}{2} \arctan\left(a^2\right) = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4},$$

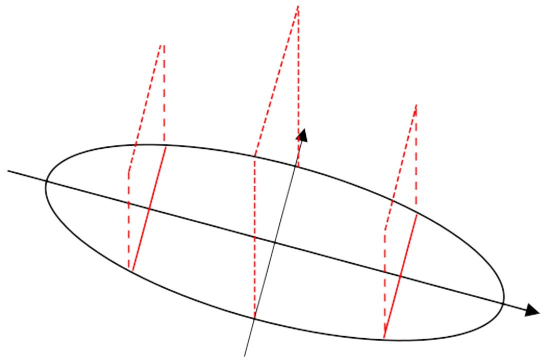
since $a^2 \to \infty$ as $a \to \infty$ and $\arctan(t) \to \dfrac{\pi}{2}$ as $t \to \infty$. Thus the area of the given region is $\dfrac{\pi}{4}$. ■

3. Sketch the solid whose base is the region enclosed by the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ and whose cross-sections perpendicular to the x-axis are squares, and find the volume of this solid. *[2]*

SOLUTION. Here is a sketch of the region and a single detached cross-section:



For most people it is hard to visualize this object or to draw it. When you try to put some cross-sections over their bases and draw this in perspective, you get something like the following:



To finish the job, you should draw the curve connecting the points where the ellipse meets the x-axis to the near upper corners of the cross-sections and then draw the curve connecting the points where the ellipse meets the x-axis to the far upper corners of the cross-sections. [My drawing program just wouldn't let me do so in a nice way.] What you get is the upper half of a sort-of "football with corner".

To find the volume of this solid, we first need to find the area of the cross-section at x. As in the solution to question **1** above, we solve the given equation for y,

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \implies \frac{y^2}{4} = 1 - \frac{x^2}{9} \implies \frac{y}{2} = \pm\sqrt{1 - \frac{x^2}{9}} \implies y = \pm 2\sqrt{1 - \frac{x^2}{9}},$$

and we can see that the region lies below $y = +2\sqrt{1 - \frac{x^2}{9}}$ and above $y = -2\sqrt{1 - \frac{x^2}{9}}$, both of which expressions make sense as long as $1 - \frac{x^2}{9} \geq 0$, *i.e.* when $-3 \leq x \leq 3$. Thus the base of the cross-section at x, for some x with $-3 \leq x \leq 3$, has length

$$2\sqrt{1 - \frac{x^2}{9}} - \left(-2\sqrt{1 - \frac{x^2}{9}}\right) = 4\sqrt{1 - \frac{x^2}{9}},$$

and, since it is a square, it follows that it has area

$$A(x) = \left(4\sqrt{1 - \frac{x^2}{9}}\right)^2 = 16\left(1 - \frac{x^2}{9}\right).$$

We can now integrate the area expression above from $x = -3$ to $x = 3$ to get the volume of the solid:

$$\begin{aligned} V &= \int_{-3}^{3} A(x)\,dx = \int_{-3}^{3} 16\left(1 - \frac{x^2}{9}\right)dx = 16\int_{-3}^{3}\left(1 - \frac{x^2}{9}\right)dx = 16\left(x - \frac{x^3}{27}\right)\Bigg|_{-3}^{3} \\ &= 16\left(3 - \frac{3^3}{27}\right) - 16\left((-3) - \frac{(-3)^3}{27}\right) = 16(3-1) - 16(-3+1) = 64 \quad \blacksquare \end{aligned}$$

[Total = 5]