Taylor Series IV - e is irrational

2021-07-26 ①

Recap: The Taylor polynomial of degree k of $f(x)$ (centred at a) is

$$T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n .$$

The remainder is

$$R_k(x) = f(x) - T_k(x).$$

(Lagrange) $R_k(x) = \dfrac{f^{(k+1)}(z)}{(k+1)!}(x-a)^{k+1}$ where z is some number strictly between a and x.

Applying all this to e^x we get (at $a=0$)

$$T_k(x) = \sum_{n=0}^{k} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!}$$

Since $e^1 = 2.718\ldots$, ~~we~~ know that $e < 3$.

$f^{(k+1)}(z)$

Thus $0 < R_k(x) = \dfrac{e^z x^{k+1}}{(k+1)!}$ for some $0 < z < x$ (when $x > 0$), so

$$0 < R_k(1) < \frac{e^1 1^{k+1}}{(k+1)!} < \frac{3}{(k+1)!} \quad \text{since } e < 3.$$

since $e^z < e^1$ since e^x is increasing

<u>Thm:</u> e is irrational, ie $e \neq \frac{a}{b}$ for any integers a & b.

<u>proof.</u> Suppose, for the sake of argument, that e is rational, ie $e = \frac{a}{b}$ for some positive integers a & b (since $e > 0$). [Otherwise multiply each by -1 first.]

$$\text{Thus } e = \frac{a}{b} = 1 + \frac{\cancel{x}\,1}{1!} + \frac{\cancel{x^2}\,1^2}{2!} + \cdots + \frac{1^k}{k!} + R_k(1)$$

so $\frac{a}{b} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{k!} + R_k(1)$

Multiply this by $k!$ on both sides, where we make sure that $k > 3 \cdot b$ (so $k > 3$ & $k > b$).

Then $\frac{k!\,a}{b} = k! + \frac{k!}{1!} + \frac{k!}{2!} + \cdots + \frac{k!}{k!} + k!\,R_k(1)$

Since $b < k$, b is a factor of $k!$, so $\frac{k!\,a}{b}$ is an integer, as are all of the terms $k!, \frac{k!}{1!}, \ldots, \frac{k!}{k!}$ on the other side

Thus $k!\,R_k(1)$ must be an integer too since it is a difference of integers

$$k!R_k(1) = \frac{k!\,a}{b} - \left(k! + \frac{k!}{1!} + \frac{k!}{2!} + \cdots + \frac{k!}{k!}\right).$$

But $R_k(1) = \frac{e^z}{(k+1)!} < \frac{e^1}{(k+1)!} < \frac{3}{(k+1)!}$

$R_k(1) > 0$ since all of $\frac{e^z}{(k+1)!}$ is > 0

$\frac{e^z}{(k+1)!}$: for some z with $0 < z < 1$

$\frac{e^z}{(k+1)!} < \frac{e^1}{(k+1)!}$: since e^x is increasing

$\frac{e^1}{(k+1)!} < \frac{3}{(k+1)!}$: since $e < 3$

IE $\quad 0 < R_k(1) < \frac{3}{(k+1)!}$

$$\Rightarrow \underset{0}{\underset{\parallel}{0 \cdot k!}} < k!\,R_k(1) < k! \cdot \frac{3}{(k+1)!} = \underbrace{\frac{3}{k+1} < 1}_{\text{since } k+1 > k > 3}$$

Thus $k!R_k(1)$ is an integer (strictly!) between 0 & 1...

Since there are no such integers and there would have to be if e was rational (by the argument above), we have a contradiction. Thus the assumption that e is rational must be incorrect.

$\therefore$ e is irrational. //