Taylor Series II - Examples and Shortcuts

(§11.10 in the textbook continued)

Taylor's Formula: the Taylor series of $f(x)$ at a (or centred at a) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

$f^{(n)}(a)$ is the n^{th} derivative of $f(x)$ evaluated at a

With rare exceptions $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ when the series converges. This allows you to expand functions in terms of powers of $(x-a)$, usually we'll have $a=0$, so really in terms of powers of x.

Last time we showed that the Taylor series of $\sin(x)$ at 0 was

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

More examples:

1° The exponential series, is the Taylor series at 0 of e^x.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^x	$e^0 = 1$
1	e^x	1
2	e^x	1
$\vdots$	$\vdots$	$\vdots$

Thus every $f^{(n)}(0) = 1$, so the Taylor series of e^x at 0 is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

This series converges for all x: Apply the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right|$$

$$= \lim_{n\to\infty} \frac{|x|}{n+1} \begin{matrix} \to |x| \\ \to \infty \end{matrix} = 0 < 1$$

so the series converges for all values of x.

[The radius of convergence is $R = \infty$ & the interval of convergence is $(-\infty, \infty)$.]

2° The Taylor series of $\ln(x)$ at $a=1$.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln(x)$	$\ln(1)=0$
1	$\frac{1}{x}$	$\frac{1}{1}=1$
2	$\frac{-1}{x^2}=-x^{-2}$	-1
3	$-(-2)x^{-3}=\frac{2}{x^3}$	2
4	$-\frac{6}{x^4}$	-6
5	$\frac{24}{x^5}$	24
6	$\frac{-120}{x^6}$	-120
⋮	⋮	⋮
k	$\frac{(-1)^{k+1}(k-1)!}{x^k}$	$(-1)^{k+1}(k-1)!$
⋮	⋮	⋮

So the Taylor series of $\ln(x)$ at $a=1$ is

$$0+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n-1)!}{n!}(x-1)^n$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(x-1)^n$$

(Recall that we previously expanded $\ln(1+x)$ as

$$\sum_{n=0}^{\infty}\underline{\quad}\,x^n$$

$\leftarrow$ look it up.)

3° The Taylor series of $f(x) = x\cos(x)$ at $a = 0$ (4)

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$x\cos(x)$	$0 \cdot 1 = 0$
1	$\cos(x) - x\sin(x)$	$1 - 0 \cdot 0 = 1$
2	$-\sin(x) - \sin(x) - x\cos(x)$	0
3	$-2\cos(x) - \cos(x) + x\sin(x)$	-3
4	$3\sin(x) + \sin(x) + x\cos(x)$	0
5	$4\cos(x) + \cos(x) - x\sin(x)$	5
$\vdots$		$\vdots$
n		0 if n is even
		$\pm n$ if n is odd
		$+$ if $n = 4k+1$
		$-$ if $n = 4k+3$

The Taylor series of $x\cos(x)$ at 0

is
$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}(2n+1)$$
$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}}$$

There is a short cut since we already know that the Taylor series of $\sin(x)$ at 0 is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, so $\cos(x) = \frac{d}{dx}\sin(x)$ has the Taylor series at 0 as

$$\sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}(2n+1)}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

& so the Taylor series of $x\cos(x)$ at 0 is

$$x \cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}.$$

Facts: 1° We can differentiate & integrat Taylor series term-by-term to get the Taylor series of the derivative or anti-derivative, resp., of the function.

2° The Taylor series of any polynomial is just the polynomial itself.

es $f(x) = x^2 + 13x + 41$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$x^2+13x+41$	41
1	$2x+13$	13
2	2	2
3	0	0
⋮	⋮	⋮

So the Taylor series of this polynomial is

$$\frac{41}{0!} + \frac{13}{1!}x + \frac{2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{0}{4!}x^4 + \cdots$$

$$= 41 + 13x + x^2 = f(x)$$

3° The sum, difference, product, ratio of the Taylor series of $f(x)$ & $g(x)$ give the Taylor series of

$f(x)+g(x)$, $f(x)-g(x)$, $f(x)g(x)$, and $f(x)/g(x)$, resp.

[In the case of $f(x)/g(x)$, you'll have to divide one Taylor series into the other algebraically – usually a total mess.]