Taylor Series, or, how to expand arbitrary functions as power series

(§11.10 in the textbook)

We are going to engineer the relevant formula from assuming that $f(x)$ can be expanded as a power series centred at a.

Suppose $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$, when the power series converges

$f^{(0)}(x)$ ← 0^{th} derivative

Within the radius of convergence about a (ie for $a-R<x<a+R$) we can differentiate both sides safely.

Then $f'(x) = f^{(1)}(x)$ ← 1^{st} derivative

$$= \frac{d}{dx} \sum_{n=0}^{\infty} a_n (x-a)^n = \sum_{n=0}^{\infty} a_n n \cdot (x-a)^{n-1}$$

$$= \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$$

will converge with same centre & radius of cvgce.

2^{nd} derivative

and $f''(x) = f^{(2)}(x) = \frac{d}{dx}\left(\sum_{n=1}^{\infty} n a_n (x-a)^{n-1}\right) = \sum_{n=1}^{\infty} a_n \cdot n(n-1)(x-a)^{n-2}$

$$= \sum_{n=2}^{\infty} n(n-1) a_n (x-a)^{n-2}$$

and $f'''(x) = f^{(3)}(x) = \frac{d}{dx}\left(\sum_{n=2}^{\infty} n(n-1)a_n(x-a)^{n-2}\right)$ ②

$$= \sum_{n=3}^{\infty} n(n-1)(n-2)a_n(x-a)^{n-3}$$

$\vdots$

In general,

$$f^{(k)}(x) = \frac{d^k}{dx^k}f(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)a_n(x-a)^{n-k}$$

When we plug in $x=a$, this tells us that

$$f^{(k)}(a) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)a_n \underbrace{(a-a)^{n-k}}_{0^{n-k}}$$

all terms except the constant term $(n=k)$ are 0 . . .

$$= k(k-1)(k-2)\cdots\underbrace{(k-k+1)}_{1}a_k$$

$$= k!\,a_k$$

so $a_k = f^{(k)}(a)/k!$,

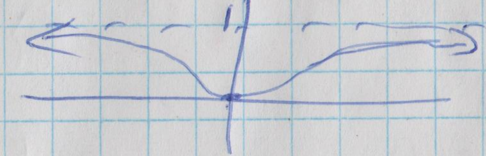
which gives us Taylor's formula for a power series for $f(x)$ centred at a:

$$f(x) \sim \sum_{k=0}^{\infty} a_k (x-a)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Warning: It's possible, though extremely rare in practice, for the series to converge near a but is equal to $f(x)$ only when $x=a$.

eg (Cauchy's annoying example)

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



has every derivative at $a=0$ defined and every derivative is 0 at $x=0$, so it's Taylor series is 0.

This is very unlikely to occur with any function (4)
you are likely to see in real life.

Most of the time a function will be
(Almost all!) equal to its Taylor series ~~at~~
when the Taylor series converge.

<u>Terminology</u>: A Taylor series centred at 0
is often called a MacLaurin series.

<u>eg</u> Let's find a Taylor series (at 0) for $\sin(x)$:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin(x)$	$\sin(0) = 0$
1	$\cos(x)$	$\cos(0) = 1$
2	$-\sin(x)$	$-\sin(0) = 0$
3	$-\cos(x)$	$-\cos(0) = -1$
4	$\sin(x)$	$\sin(0) = 0$
5	$\cos(x)$	$\cos(0) = 1$
$\vdots$	$\vdots$	$\vdots$

so $f^{(n)}(0)$ is 0 when n is even
& $f^{(n)}(0)$ is alternating between -1 & $+1$ when n is odd.

Thus the Taylor series of $\sin(x)$ at $a=0$

is $0 + \frac{1x}{1!} + \frac{0 \cdot x^2}{2!} + \frac{(-1)x^3}{3!} + \frac{0 \cdot x^4}{4!} + \frac{1x^5}{5!} + \frac{0x^6}{6!} + \frac{(-1)x^7}{7!} + \cdots$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$(2k+3)! = (2k+3)(2k+2)\underbrace{(2k+1)\cdots 2 \cdot 1}_{(2k+1)!}$

This series has radius of convergence $R=\infty$:
Use the Ratio Test:

$$\lim_{k\to\infty} \left| \frac{(-1)^{k+1} x^{2k+3} / (2k+3)!}{(-1)^k x^{2k+1} / (2k+1)!} \right| = \lim_{k\to\infty} \left| \frac{(-1)^{k+1} x^{2k+3}}{(2k+3)!} \cdot \frac{(2k+1)!}{(-1)^k x^{2k+1}} \right|$$

$$= \lim_{k\to\infty} \left| \frac{(-1)x^2}{(2k+3)(2k+2)} \right| = \lim_{k\to\infty} \frac{x^2 \to x^2}{(2k+3)(2k+2) \to \infty} = 0 < 1$$

so the series converges absolutely for all x.

So $\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} - \cdots$$

for all x.

We can use initial chunks of this series to compute approximations to $\sin(x)$.