Series VIII - The Ratio and Root Tests

(Section 11.7 in the textbook.)

Ratio Test Suppose $\sum_{n=0}^{\infty} a_n$ is some series and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ for some real number L.

Then: (1) If $L < 1$, the series converges absolutely.

(2) If $L > 1$, the series diverges.

(3) If $L = 1$, the series might converge or diverge. [The Ratio Test tells us nothing...]

Why does this work? If $L < 1$, then eventually $\left|\frac{a_{n+1}}{a_n}\right| < L + \varepsilon$ for all n past that point for some $\varepsilon > 0$ such that $L + \varepsilon < 1$, so we can compare $|a_n|$ to $K(L+\varepsilon)^n$, and $\sum K(L+\varepsilon)^n$ converges since it is a geometric series with $|r| = L + \varepsilon < 1$.

Similarly, if $L > 1$ then eventually $\left|\frac{a_{n+1}}{a_n}\right| > L + \varepsilon$ for some $\varepsilon > 0$ s.t. $L+\varepsilon > 1$ and all n past that point, so the series can be compared to $\sum k(L+\varepsilon)^n$ which diverges because it is a geometric series with $|r| = L+\varepsilon > 1$.

If $L = 1$, we can't compare it conveniently to a geometric series, so we fall back on other tests.

<u>Note:</u> Our main use for this test will be for power series, series of the form $\sum_{n=0}^{\infty} a_n x^n$, coming soon in §11.8 of the text. Our other tests will be to deal with any specific values of x which are left undealt with by the Ration and Root Tests.

eg $\sum_{n=0}^{\infty} \frac{n}{2^n}$ This is a bit messy, at best, if you use other tests, but is quick with the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} \right| = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \lim_{n\to\infty} \frac{1}{2} \cdot \frac{n+1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{2} \cdot \Big(1 + \underbrace{\frac{1}{n}}_{\to 0}\Big) = \frac{1}{2}(1+0) = \frac{1}{2} < 1,$$

so the series converges by the Ratio Test.

eg $\sum_{n=0}^{\infty} \frac{n!}{2^n}$ Ratio Test: $\lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{2^{n+1}}}{\frac{n!}{2^n}} \right| = \lim_{n\to\infty} \frac{(n+1)!}{2^{n+1}} \cdot \frac{2^n}{n!}$

$n! = n(n-1)(n-2)\cdot \ldots \cdot 2 \cdot 1$

$(n+1)! = (n+1)\cdot n!$

$$= \lim_{n\to\infty} \frac{1}{2} \cdot (n+1) = \infty > 1, \text{ so}$$

the series diverges by the Ratio Test.

<u>Note</u>: The Ratio Test tends to work well if a_n is given by a formula defined (or definable) using only multiplication & division.

eg For which values of x does $\sum_{n=0}^{\infty} \frac{x^n}{n}$ converge?

First, throw the Root Test at this problem:

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\cdot\frac{n}{n+1}\right| = \lim_{n\to\infty}\left|x\cdot\frac{n}{n+1}\right|$$

$$= \lim_{n\to\infty}\frac{n}{n+1}\cdot|x| = |x|\cdot\lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= |x|\lim_{n\to\infty}\frac{1}{1+\underbrace{\frac{1}{n}}_{\to 0}} = |x|\cdot 1 = |x|.$$

This tells us, by the Root Test, that the given series converges if $|x|<1$, ie $-1<x<1$, and diverges if $|x|>1$, ie $x<-1$ or $x>1$. Unfortunately, it tells us nothing if $|x|=1$, ie $x=-1$ or $x=1$.

For $x=\pm 1$, we haul out our other tests:

$x=-1$: $\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\cdots$ converges by the Alternating Series Test:

(1) it alternates

(2) $\left|\frac{(-1)^{n+1}}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n} = \left|\overbrace{\frac{(-1)^n}{n}}^{a_n}\right|$

& (3) $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{1}{n} = 0$.

$x = 1$: $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges by the p-Test, since $p = 1 \leq 1$.

eg $\sum_{n=1}^{\infty} \frac{1}{n^n} = 1 + \frac{1}{4} + \frac{1}{27} + \frac{1}{256} + \cdots$

We can easily check this converges by the Basic Comparison Test: $\frac{1}{n^n} \leq \frac{1}{n^2}$ for $n \geq 2$ & $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-Test since it has $p = 2 > 1$.

If we try to throw the Ratio Test at this we have to compute $\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)^{n+1}}}{\frac{1}{n^n}} \right| = \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}}$

which is doable, but you have to work <u>much</u> harder than [ ].

<u>Moral:</u> The Ratio Test is a powerful tool, but some of the time other tools are easier to use.

Root Test: (Theoretically a little stronger than the Ratio Test, but tends to give harder limits to compute for most series.)

Suppose we are given a series $\sum_{n=0}^{\infty} a_n$, and $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = L$. Then

(1) If $L < 1$, the series converges absolutely.
(2) If $L > 1$, the series diverges.
(3) If $L = 1$, the series may converge or diverge. [The test tells us nothing.]

This test tends to work well for series where a_n is given by some formula involving n^{th} powers glued together by multiplication & division only. [Not so good for factorials, usually.]

eg Let's revisit $\sum_{n=1}^{\infty} \frac{1}{n^n}$ with the Root Test.

(hard with the Ratio Test)

$$\lim_{n\to\infty} \left|\frac{1}{n^n}\right|^{1/n} = \lim_{n\to\infty} \left[\left(\frac{1}{n}\right)^n\right]^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n}\right)^{\cancel{n}\cdot\frac{1}{\cancel{n}}}$$

$$= \lim_{n\to\infty} \frac{1}{n} = 0 < 1,$$

so the series converges by the Root Test...

eg Let's revisit $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ (easy with the Ratio Test)

$$\lim_{n\to\infty} \left|\frac{2^n}{n!}\right|^{1/n} = \lim_{n\to\infty} \frac{(2^n)^{1/n}}{(n!)^{1/n}} = \lim_{n\to\infty} \frac{2}{(n!)^{1/n}} \longrightarrow ?$$

This limit is ~~pretty~~ very hard to compute without using something like Stirling's Formula: $n! \sim \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n$

ie $\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1,$

& even with Stirling's Formula it's pretty hard...