Series VI - The Alternating Series Test

(§11.4 in the textbook)

So far we have several tests for convergence of series, but all (except for the Divergence Test) really only work for series of all positive terms (or all negative after ~~multiplying~~ multiplying by -1).

What about a series like

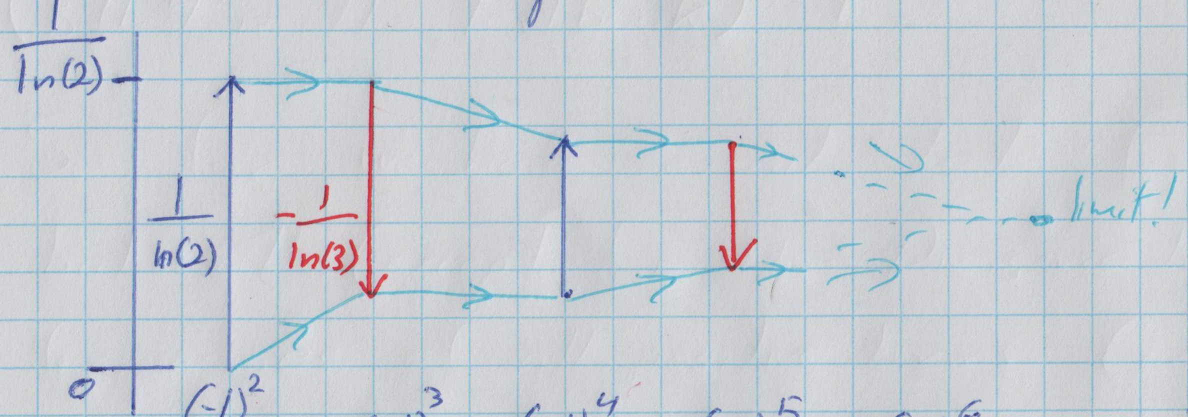
$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)} = \frac{1}{\ln(2)} - \frac{1}{\ln(3)} + \frac{1}{\ln(4)} - \frac{1}{\ln(5)} + \cdots ?$$

A series where the non-zero terms alternate sign, is an <u>alternating series</u>.

The corresponding series of positive terms does <u>not</u> converge in this case.

Observe that since $x > \ln(x)$ for all $x \geq 1$, we have that $\frac{1}{\ln(n)} > \frac{1}{n}$. Since $\sum_{n=2}^{\infty} \frac{1}{n}$, the harmonic series, diverges, so does $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$.

Look at the partial sums



Since $\ln(x)$ is an increasing function, $\frac{1}{\ln(x)}$ is a decreasing function.

$$\frac{(-1)^2}{\ln(2)} + \frac{(-1)^3}{\ln(3)} + \frac{(-1)^4}{\ln(4)} + \frac{(-1)^5}{\ln(5)} + \frac{(-1)^6}{\ln(6)} + \cdots$$

$$= \frac{1}{\ln(2)} - \frac{1}{\ln(3)} + \frac{1}{\ln(4)} - \frac{1}{\ln(5)} + \frac{1}{\ln(6)} - \cdots$$

Observe that each partial sum is going to be within $\left|\frac{(-1)^n}{\ln(n)}\right| = \frac{1}{\ln(n)}$ of all of the subsequent partial sums. Since $\lim_{n\to\infty} \frac{1 \to 1}{\ln(n) \to \infty} = 0$, this means the partial have to home in on some real number.

By definition, this means that $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges.

The Alternating Series Test

Suppose $\sum_{n=0}^{\infty} a_n$ is a series such that

(0) the series alternates sign (if you ignore any zero terms) eventually, ie past some point;

(1) $|a_{n+1}| \leq |a_n|$ past some (possibly different) point;

and (2) $\lim_{n \to \infty} |a_n| = 0$ [ie it survives the Divergence Test].

Then the series $\sum_{n=0}^{\infty} a_n$ converges.

Terminology: A series that converges, but the series of absolute values $\left(\sum_{n=0}^{\infty} |a_n|\right)$ diverges, $\left(\sum_{n=0}^{\infty} a_n\right)$ is said to converge conditionally.

If the series of absolute values converges too, the original series is said to converge absolutely.

eg 1° The prototype for alternating series is the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

This converges by the Alternating Series Test because

(0) $\frac{(-1)^{n-1}}{n}$ alternates sign because $(-1)^{n-1}$ alternates sign and $\frac{1}{n} > 0$,

(1) $\left| \frac{(-1)^{n+1-1}}{n+1} \right| = \frac{1}{n+1} \leq \frac{1}{n} = \left| \frac{(-1)^{n-1}}{n} \right|$ for all $n \geq 1$, and

(2) $\lim_{n\to\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \lim_{n\to\infty} \frac{1}{n} = 0$.

2° What about $\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \cdots$?

This alternates (from the beginning), $|(-1)^{n+1}| = 1 \leq 1 = |(-1)^n|$ (from the beginning), but $\lim_{n\to\infty} |(-1)^n| = \lim_{n\to\infty} 1 = 1 \neq 0$ so the Alt. Series Test tells us nothing. [But the fact that the series fails the Divergence Test, tells us it diverges.]

3° $\displaystyle\sum_{n=\phi}^{\infty} \frac{\cos(n\pi)}{ne^n} = \frac{-1}{1e^1} + \frac{1}{2e^2} + \frac{-1}{3e^3} + \frac{1}{4e^4} + \frac{-1}{5e^5} + \cdots$

$\displaystyle = \frac{-1}{e} + \frac{1}{2e^2} - \frac{1}{3e^3} + \frac{1}{4e^4} + \frac{-1}{5e^3} + \cdots$

Observe that

$\cos(n\pi) = (-1)^n$,

so alternation happens since $ne^n > 0$ for all $n \geq 1$.

(0) Alternation ✓

(1) $\cancel{\left|\frac{\cos(n\pi)}{n}\right|}$

$$\left|\frac{\cos((n+1)\pi)}{(n+1)e^{n+1}}\right| = \frac{1}{(n+1)e^{n+1}} < \frac{1}{ne^n} = \left|\frac{\cos(n\pi)}{ne^n}\right|$$

since $(n+1)e^{n+1} > ne^n$. ✓

(2) $\displaystyle\lim_{n\to\infty} \left|\frac{\cos(n\pi)}{ne^n}\right| = \lim_{n\to\infty} \frac{\overset{\to 1}{1}}{\underset{\to\infty}{ne^n}} = 0$ ✓

Thus the series converges by the Alternating Series Test. Does it converge conditionally or absolutely?

i.e. Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{ne^n}$ converge?

Observe that if $n \geq 1$, then $0 \leq \frac{1}{ne^n} \leq \frac{1}{e^n} = \left(\frac{1}{e}\right)^n$ (6)

so it converges using the Basic Comparison Test

by comparison with the geometric series

$$\sum_{n=0}^{\infty} \frac{1}{e^n} = \sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$$ which converges because $|r| = \frac{1}{e} < 1$.

$\therefore \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{ne^n}$ converges absolutely.

Next time: more on absolute vs. conditional convergence