Series IV - The (Basic) Comparison Test

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(§11.5 in the text - we'll go back to do §11.4 afterwards.)

Quick recap of series so far:

$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$ **converges** ("adds up sensibly")

exactly when $\lim_{k \to \infty} \left(\sum_{n=0}^{k} a_n \right) = \lim_{k \to \infty} (a_0 + a_1 + a_2 + \cdots + a_k)$

exists (and then the limit is the sum). If

the limit fails to exist the series **diverges**.

We have various tests for checking if a series converges

0° Series that are easy to find the sum of:

a) geometric series: $a + ar + ar^2 + ar^3 + \cdots$,

which converge to $\frac{a}{1-r}$ if $a = 0$ or if $|r| < 1$

and which diverges otherwise.

b) telescoping series: series in which (possibly after some

algebra) each term cancels out the next past some point, 1° Divergence Test: $\lim_{n\to\infty} a_n \neq 0$

$\Rightarrow \sum_{n=0}^{\infty} a_n$ diverges.

It's not foolproof: there are series for which $\lim_{n\to\infty} a_n = 0$ but the series diverges.

(eg $a_n = \frac{1}{n}$)

2° Integral Test: If $a_n = f(n)$ for some nice continuous function $f(x)$ on $[k, \infty)$ (s.t. $f(x) > 0$ on $[k, \infty)$), then $\sum_{n=k}^{\infty} a_n$ converges or diverges exactly as the improper integral $\int_k^{\infty} f(x)\,dx$ converges or diverges.

Limitations: need to be able to find and integrate a suitable function; works only on series which are eventually all positive (or all negative by multiplying by -1).

3° p-Test: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ & diverges if $p \leq 1$. (p is a real number)

4° <u>(Basic) Comparison Test</u>

Suppose that $0 \leq a_n \leq b_n$ for all n past some point. Then (1) If $\sum_{n=k}^{\infty} b_n$ converges, then so does $\sum_{n=k}^{\infty} a_n$;

(2) If $\sum_{n=k}^{\infty} a_n$ diverges, then so does $\sum_{n=k}^{\infty} b_n$.

<u>Limitations</u>: a) You have to be able to guess whether the series you're interested in is convergent or divergent, and then find a suitable series to compare it to.

b) Only works for series that are eventually entirely non-negative (or entirely non-positive by multiplying by -1).

eg $\sum_{n=3}^{\infty} \frac{1}{n^2-3n+1}$. Note that $\frac{1}{n^2-3n+1} > 0$ for $n \geq 3$.

This series ought to converge because $\sum_{n=3}^{\infty} \frac{1}{n^2}$ does by the p-Test since $p=2>1$, and when n is large $n^2-3n+1 \approx n^2$.

Since $n^2 > n^2-3n+1$, (since $-3n+1$ will be negative)

we have that $\frac{1}{n^2} < \frac{1}{n^2-3n+1}$, but this is not useful here since we need to compare it to something <u>bigger</u> that still converges.

Morals:
1) Have to get the direction right.
2) May have to fiddle a suitable comparison series.

$n^2-3n+1 > n^2-6n+9$ once $n \geq 4$

Since $n^2-6n+9 = (n-3)^2$

we have $0 < \frac{1}{n^2-3n+1} < \frac{1}{(n-3)^2}$ so because

$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, $\sum_{n=3}^{\infty} \frac{1}{n^2-3n+1} = 1 + \sum_{n=4}^{\infty} \frac{1}{n^2-3n+1}$ converges too.

Not all are this hard, fortunately:

eg $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n+1}$

all $\overset{\vee}{0}$

Since $n^2+3n+1 > n^2$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (by the p-Test), the given series converges by the Comparison Test.

eg $\displaystyle\sum_{n=0}^{\infty} \frac{|\cos(n)|}{2^n}$.

a) $\dfrac{|\cos(n)|}{2^n} \geq 0$

b) $\dfrac{|\cos(n)|}{2^n} \leq \dfrac{1}{2^n}$

The series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ is a geometric

series with common ratio $r = \frac{1}{2}$, so $|r| = |\frac{1}{2}| = \frac{1}{2} < 1$, converges. By the Comparison Test, it follows that $\displaystyle\sum_{n=0}^{\infty} \frac{|\cos(n)|}{2^n}$ converges, too.

eg. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

[Integral Test works well here too.]

Notice that

$$\lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{x\to\infty} \frac{\ln(x)}{x} \begin{matrix}\to\infty \\ \to\infty\end{matrix}$$

Apply l'Hôpital's Rule.

$$= \lim_{x\to\infty} \frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

so this passes the Divergence Test.

But if $n \geq 1$, $0 \leq \frac{\ln(n)}{n}$, and even more, if $n \geq 3$,

$$\frac{1}{n} < \frac{\ln(n)}{n}$$

since $\ln(3) > 1$ (and $\ln(x)$ is increasing).

It follows by the Comparison Test, that since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ diverges as well.

Next time: Limit Comparison Test (not in the textbook)