2021-07-12 ①

Series III - The Integral Test, [for convergence of series]

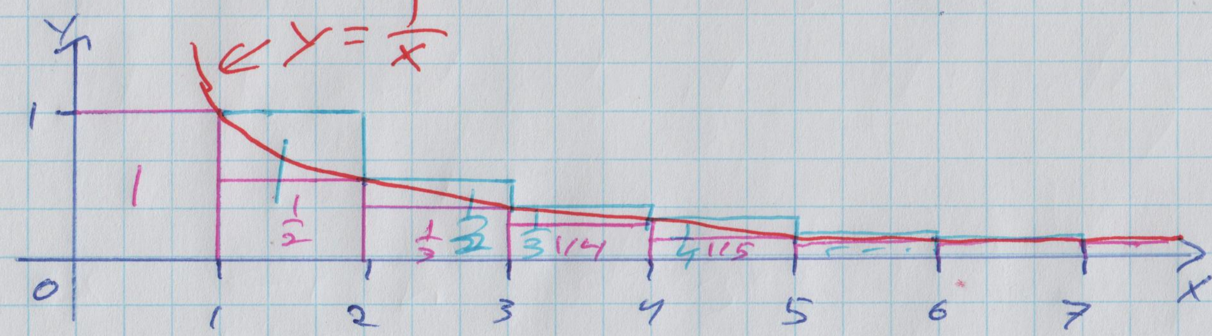
(§11.3 in the text) or, why we really did improper integrals.

We'll illustrate the idea behind the Integral Test by checking that 1° $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges [we already did this in a different way]

& 2° $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots$ converges.

We'll turn convergence into area problems...

1°



The areas of the rectangles ~~sum to~~ are the sequence being summed.

Note that the rectangles (from the second one on) fit under $y = \frac{1}{x}$.
If we shove them over by one, then the area under $y = \frac{1}{x}$, for $1 \le x < \infty$, fits inside the rectangles.

This means that

$$\int_1^\infty \frac{1}{x}\,dx < \sum_{n=1}^{\infty} \frac{1}{n}$$

$\int_1^\infty \frac{1}{x}\,dx$ " area under $y = \frac{1}{x}$

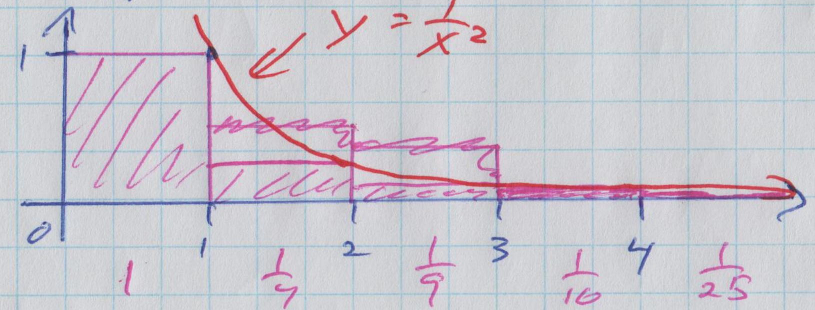
$\sum_{n=1}^{\infty} \frac{1}{n}$ " area of all the rectangles.

However, $\int_1^\infty \frac{1}{x}\,dx = \lim_{a\to\infty} \int_1^a \frac{1}{x}\,dx = \lim_{a\to\infty} \ln(x)\Big|_1^a$

$$= \lim_{a\to\infty} \big(\ln(a) - \underbrace{\ln(1)}_{0}\big) = \lim_{a\to\infty} \ln(a) = \infty$$

so the area under $y = \frac{1}{x}$, $1 \le x < \infty$, is infinite, and thus the sum of the areas of the rectangles, i.e. $\sum_{n=1}^{\infty} \frac{1}{n}$, is infinite too. Thus $\sum_{n=1}^{\infty} \frac{1}{n}$ "$= \infty$" so it diverges.

2. $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$ converges:



The term $\frac{1}{n^2}$ of the series is the area of the rectangle with base $[n-1, n]$ & height $\frac{1}{n^2}$.

All of these rectangles (except the first) fit under the graph of $y = \frac{1}{x^2}$ for $1 \le x < \infty$.

Thus the series $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2}$ computes an area contained in the area under $y = \frac{1}{x^2}$.

Thus since $\int_1^{\infty} \frac{1}{x^2}dx = \lim_{a \to \infty} \int_1^a \frac{1}{x^2}dx = \lim_{a \to \infty} \frac{-1}{x}\Big|_1^a$

$= \lim_{a \to \infty} \left(-\frac{1}{a} - \left(\frac{-1}{1}\right)\right) = \lim_{a \to \infty} \left(1 - \frac{1}{a}\right) = 1 - 0 = 1$

has a finite value, so does $\sum_{n=1}^{\infty} \frac{1}{n^2}$, so it converges.

Integral Test

Suppose $f(x) \geq 0$ and is decreasing for $x \geq k$ for some k and suppose further that $a_n = f(n)$ (for $n \geq k$). Then $\sum_{n=0}^{\infty} a_n$ converges (or diverges)

exactly when the improper integral $\int_k^{\infty} f(x)\,dx$ converges (i.e. evaluates to a real number) or diverges.

The Integral Test can be applied directly, as in the previous examples, or it can be used to obtain other "quick and dirty" convergence tests.

3° $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$ converges (or not) exactly as $\int_0^{\infty} \frac{1}{1+x^2}\,dx$ does

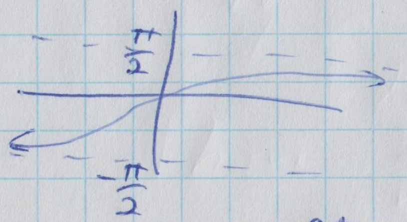
because $f(x) = \frac{1}{1+x^2} > 0$ for all x in $[0,\infty)$

and is decreasing on $[0,\infty)$ [since $1+x^2$ is increasing]

[& the numerator is constant]

and $\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{a\to\infty} \int_0^a \frac{1}{1+x^2}\,dx = \lim_{a\to\infty} \arctan(x)\Big|_0^a$

$$= \lim_{a\to\infty} \left(\arctan(a) - \underbrace{\arctan(0)}_{=0}\right)$$



$$= \lim_{a\to\infty} \arctan(a) = \frac{\pi}{2}.$$

So $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$ converges.

<u>The p-Test</u> $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$

& diverges if $p \leq 1$. (If $p \leq 0$, Divergence Test will do the job)

<u>proof:</u> By the Integral Test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges
(Assume $p>0$)
exactly when $\int_1^\infty \frac{1}{x^p}\,dx$ converges.

$$\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx = \lim_{a\to\infty} \int_1^a x^{-p}\,dx = \lim_{a\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_1^a$$

$$= \lim_{a\to\infty} \left(\frac{a^{1-p}}{1-p} - \frac{1}{1-p}\right)$$

but $a^{1-p} \to \infty$ if $p<1$ and $\to 0$ if $p>1$.

$\boxed{\text{If } p=1\text{, this fails, but we already know it diverges then.}}$

Thus $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \leq 1$

& converges if $p > 1$. //

e.g. Does $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$ converge?

Yes, by the p-Test, because for this series $p = \pi > 1$.