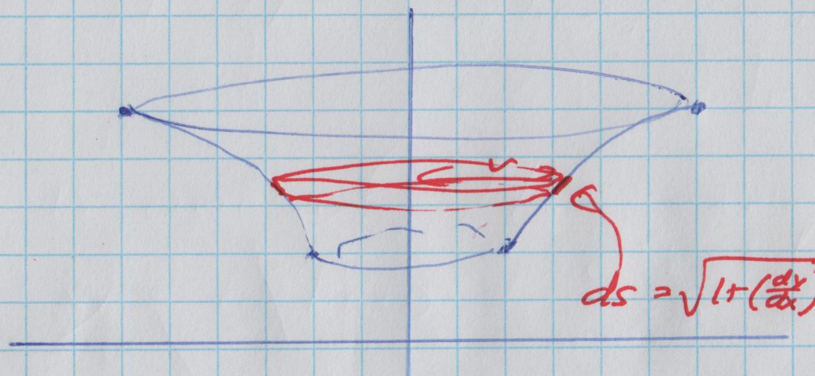
Areas of Surfaces of Revolution (§9.10 in the text)

(or, what is the surface area of a solid of revolution?)



Revolve $y = f(x)$, $a \le x \le b$, about a line to get a "surface of revolution".

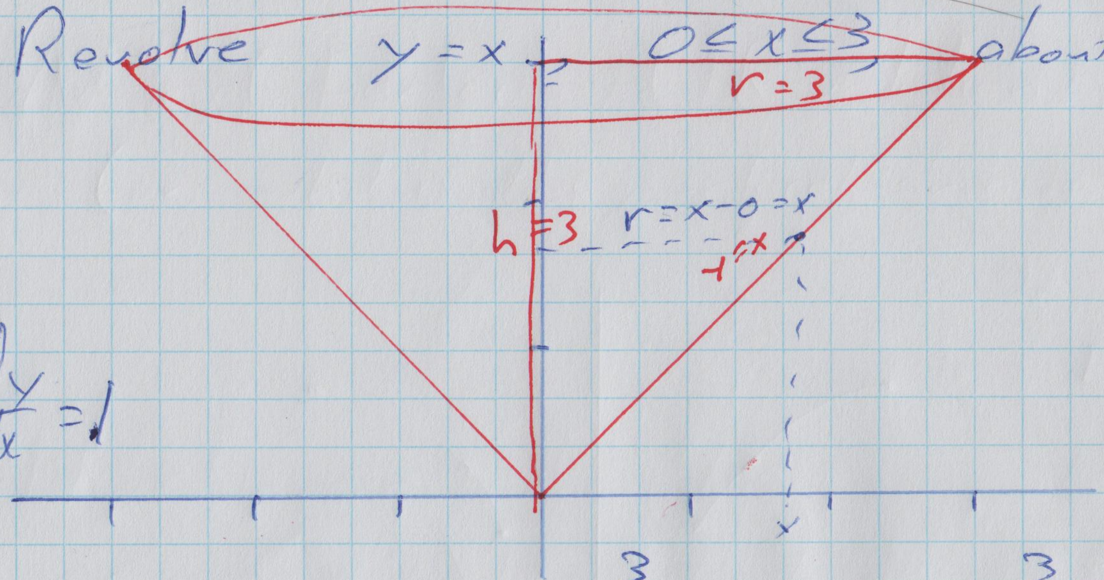
When you revolve an~~d~~ infinitesimal bit of arc-length through a radius of r, you get (approximately) ~~the~~ ~~surfa~~ a little cylinder of height ds & radius r, so it contributes an area of $2\pi r \underbrace{ds}_{h}$ to the surface.

$$\text{Area of surface} = \int_a^b 2\pi r\,ds = 2\pi \int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad \left(\begin{array}{c}\text{revolving}\\ \text{about the}\\ y\text{-axis}\end{array}\right)$$

(r=x as drawn above)

$$= 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad \left(\begin{array}{c}\text{revolving}\\ \text{about the}\\ x\text{-axis}\end{array}\right)$$

1° Revolve $y = x$, $0 \le x \le 3$, about the y-axis.



$\frac{dy}{dx} = 1$

This gives a cone with base radius $r=3$ & height 3 (upside down cone...).

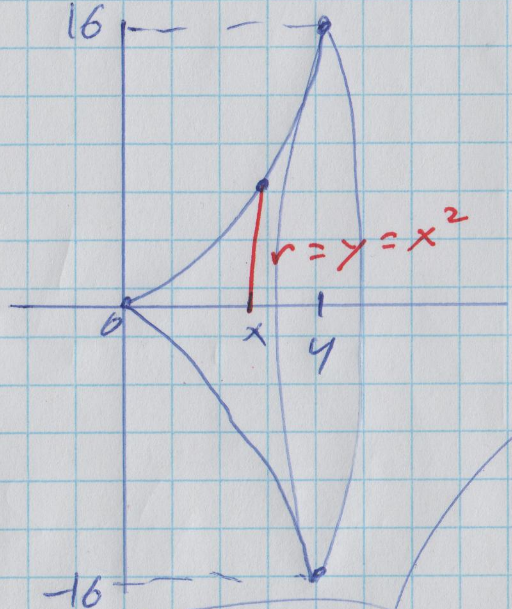
$$\text{Surface Area} = \int_0^3 2\pi r\, ds = \int_0^3 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_0^3 x\sqrt{1+1^2}\, dx = 2\pi \int_0^3 x\sqrt{2}\, dx$$

$$= \cancel{2}\pi\sqrt{2} \cdot \frac{x^2}{\cancel{2}} \Big|_0^3 = \pi\sqrt{2}\cdot 3^2 - \pi\sqrt{2}\cdot 0^2$$

$$= 9\sqrt{2}\pi - 0 = 9\sqrt{2}\pi$$

2° Revolve $y = x^2$, $0 \le x \le 4$, about the x-axis.



$$\frac{dy}{dx} = \frac{d}{dx}x^2 = 2x$$

$$\& \ (2x)^2 = 4x^2$$

$$SA = \int_0^4 2\pi r\, ds = 2\pi \int_0^4 x^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$x = \frac{1}{2}\tan(\theta)$

$2x = \tan(\theta)$

$dx = \frac{1}{2}\sec^2(\theta)d\theta$

$$= 2\pi \int_0^4 x^2 \sqrt{1+4x^2}\, dx$$

$$= 2\pi \int_{x=0}^{x=4} \frac{1}{4}\tan^2(\theta) \sqrt{\underbrace{1+4\cdot\frac{1}{4}\tan^2(\theta)}_{\sec^2(\theta)}}\ \frac{1}{2}\sec^2(\theta)\, d\theta$$

$$= \cancel{2}\pi \frac{1}{\cancel{2}4} \int_{x=0}^{x=4} \tan^2(\theta)\sec(\theta)\sec^2(\theta)\, d\theta$$

$$= \frac{\pi}{4} \int_{x=0}^{x=4} \tan^2(\theta)\sec^3(\theta)\, d\theta$$

$$= \frac{\pi}{4} \int_{x=0}^{x=4} (\cancel{4}\sec^2(\theta) - 1)\sec^3(\theta)\, d\theta$$

$$= \frac{\pi}{4}\left[\int \sec^5(\theta)\,d\theta - \int \sec^3(\theta)\,d\theta\right]\Bigg|_{x=0}^{x=4}$$

$$= \frac{\pi}{4}\left[\frac{1}{5-1}\tan(\theta)\sec^{5-2}(\theta) - \frac{5-2}{5-1}\int \sec^{5-2}(\theta)\,d\theta - \int \sec^3(\theta)\,d\theta\right]\Bigg|_{x=0}^{x=4}$$

$$= \frac{\pi}{4}\left[\frac{1}{4}\tan(\theta)\sec^3(\theta) - \frac{7}{4}\int \sec^3(\theta)\,d\theta\right]\Bigg|_{x=0}^{x=4}$$

$$= \frac{\pi}{16}\left[\tan(\theta)\sec^3(\theta) - 7\left(\frac{1}{2}\tan(\theta)\sec(\theta) - \frac{1}{2}\int \sec(\theta)\,d\theta\right)\right]\Bigg|_{x=0}^{x=4}$$

$$= \frac{\pi}{16}\left[\tan(\theta)\sec^3(\theta) - \frac{7}{2}\tan(\theta)\sec(\theta) + \frac{7}{2}\ln(\sec(\theta)+\tan(\theta))\right]\Bigg|_{x=0}^{x=4}$$

but $\sec(\theta) = \sqrt{1+\tan^2(\theta)} = \sqrt{1+(2x)^2} = \sqrt{1+4x^2}$ & $\tan(\theta) = 2x$

$$= \frac{\pi}{16}\left[2x\left(1+4x^2\right)^{3/2} - \frac{7}{\cancel{2}}\cancel{2}x\left(1+4x^2\right)^{1/2} + \frac{7}{2}\ln\left(2x+\left(1+4x^2\right)^{1/2}\right)\right]\Bigg|_{x=0}^{x=4}$$

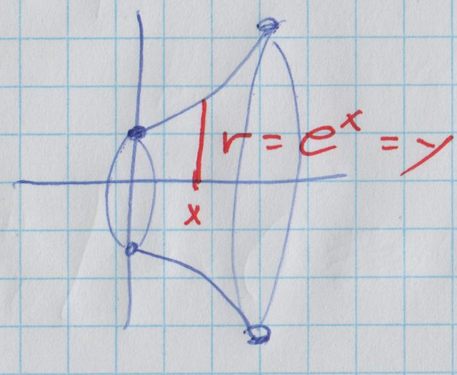
$$= \frac{\pi}{16}\left[2\cdot 4\cdot(\sqrt{17})^3 - 7\cdot 4\cdot\sqrt{17} + \frac{7}{2}\ln\left(4+\sqrt{17}\right)\right]$$

$$- \frac{\pi}{16}\left[\cancel{2\cdot 0\cdot 1^{3/2}} - \cancel{7\cdot 0\cdot 1^{1/2}} + \frac{7}{2}\underbrace{\ln\left(2\cdot 0+1^{1/2}\right)}_{=\ln(1)=0}\right]$$

$$= \frac{\pi}{16}\left[8\cdot 17\cdot\sqrt{17} - 28\sqrt{17} + \frac{7}{2}\ln\left(4+\sqrt{17}\right)\right]$$

$= \circ\circ\circ$ Ugly...

3° Revolve $y = e^x$, $0 \le x \le \ln(3)$, about the x-axis.



$$SA = \int_0^{\ln(3)} 2\pi r\, ds = \int_0^{\ln(3)} 2\pi e^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\ln(3)} 2\pi e^x \sqrt{1+(e^x)^2}\, dx$$

$u = e^x \quad du = e^x dx$

x	u
0	0
$\ln(3)$	3

$$= \int_0^3 2\pi \underbrace{\sqrt{1+u^2}}_{\sec(\theta)}\, du$$

$u = \tan(\theta)$

$du = \sec^2(\theta)\,d\theta$

$\sec(\theta) = \sqrt{1+u^2}$

$$= 2\pi \int_{u=0}^{u=3} \sec(\theta)\sec^2(\theta)\,d\theta = 2\pi \int_{u=0}^{u=3} \sec^3(\theta)\,d\theta$$

$$= \cancel{2}\pi \left(\frac{1}{\cancel{2}} \sec(\theta)\tan(\theta) - \frac{1}{\cancel{2}}\int \sec(\theta)\,d\theta \right) \Big|_{u=0}^{u=3}$$

$$= \pi \left(\sec(\theta)\tan(\theta) - \ln(\sec(\theta) + \tan(\theta)) \right) \Big|_{u=0}^{u=3}$$

$$= \pi \left(u\sqrt{1+u^2} - \ln\left(u + \sqrt{1+u^2}\right) \right) \Big|_{u=0}^{u=3}$$

$$= \pi \left(3\sqrt{10} - \ln(3+\sqrt{10}) \right) - \cancel{\pi \left(0\cdot 1 - \ln(0+\sqrt{1}) \right)}$$

$$= \pi \left(3\sqrt{10} - \ln(3+\sqrt{10}) \right).$$

If you ever get a nice number from a non-trivial arc-length or area of a surface of revolution problem, somebody worked hard to make it so.