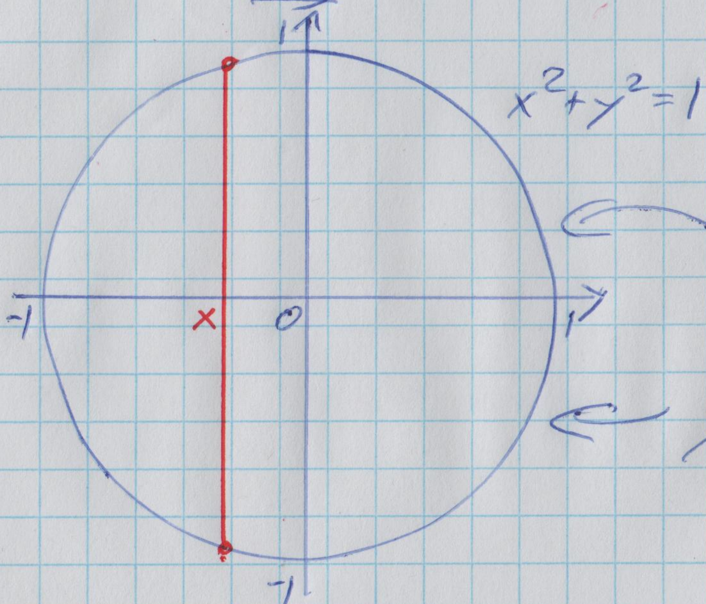
Areas (of regions between curves) [§9.1 in the textbook]

A lot of shapes can be conveniently described by falling between two curves.

eg unit circle centred at the origin



$y=\sqrt{1-x^2}$

$y=-\sqrt{1-x^2}$

Integrate vertical cross-sections (ie their lengths) to compute the area.

"Lengths of cross-sections sweep out (or add up to) areas."

$$\text{Area}=\int_a^b \underbrace{(\text{upper}-\text{lower})}_{\text{length}}\,dx$$

$$\text{Area}=\int_{-1}^{1}\left(\sqrt{1-x^2}-\left[-\sqrt{1-x^2}\right]\right)dx$$

$$=2\int_{-1}^{1}\sqrt{1-x^2}\,dx$$

$x=\sin(\theta)$

$dx=\cos(\theta)d\theta$

x	θ
-1	$-\frac{\pi}{2}$
1	$\frac{\pi}{2}$

$$=2\int_{\pi/2}^{\pi/2}\sqrt{1-\sin^2(\theta)}\cdot\cos(\theta)d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} \sqrt{\cos^2(\theta)}\cdot\cos(\theta)\,d\theta = 2\int_{-\pi/2}^{\pi/2} \cos(\theta)\cos(\theta)\,d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} \cos^2(\theta)\,d\theta = 2\int_{-\pi/2}^{\pi/2} \left(\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right)d\theta$$

$\cancel{2}\alpha = 2\theta$
$d\alpha = 2d\theta$
$d\theta = \frac{1}{2}d\alpha$

$$= \cancel{2}\cdot\frac{1}{\cancel{2}}\int_{-\pi}^{\pi} (1+\cos(\alpha))\cdot\frac{1}{2}\,d\alpha$$

θ	α
$-\frac{\pi}{2}$	$-\pi$
$\frac{\pi}{2}$	π

$$= \frac{1}{2}\left(\alpha+\sin(\alpha)\right)\Big|_{-\pi}^{\pi}$$

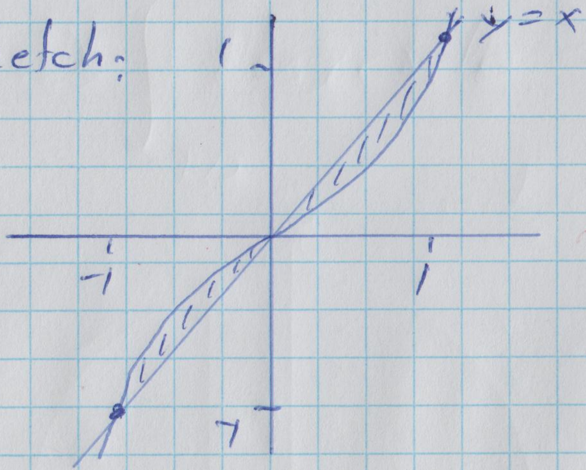
$$= \frac{1}{2}\left(\pi+\overset{0}{\overset{\shortparallel}{\sin(\pi)}}\right) - \frac{1}{2}\left(-\pi+\overset{0}{\overset{\shortparallel}{\sin(-\pi)}}\right)$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

This is correct since the area of a circle of radius 1 is $\pi\cdot 1^2 = \pi$.

Example: Find the area of the region between $y = x^3$ and $y = x$, for $-1 \le x \le 1$.

Sketch:



It's useful to sketch the region to sort out which function is upper and which is lower. In this case, $y = x^3$ is above $y = x$ for $-1 \le x \le 0$, and below for $0 \le x \le 1$.

$$\text{Area} = \int_{-1}^{1} (\text{upper} - \text{lower})\, dx = \int_{-1}^{0} (x^3 - x)\, dx + \int_{0}^{1} (x - x^3)\, dx$$

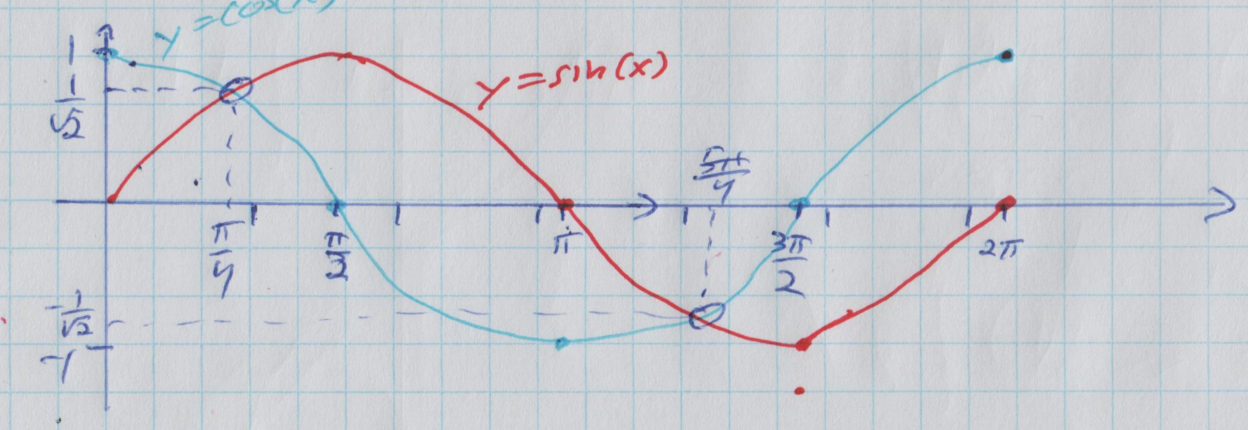
$$= \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_{-1}^{0} + \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_{0}^{1}$$

$$= \cancel{\left(\frac{0^4}{4} - \frac{0^2}{2}\right)} - \left(\frac{(-1)^4}{4} - \frac{(-1)^2}{4}\right) + \left(\frac{1^2}{2} - \frac{1^2}{4}\right) - \cancel{\left(\frac{0^2}{2} - \frac{0^4}{4}\right)}$$

$$= -\left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) = -\left(-\frac{1}{4}\right) + \frac{1}{4} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Another example: Find the area between $y=\cos(x)$ and $y=\sin(x)$ for $0 \le x \le 2\pi$.

Sketch:



$$\text{Area} = \int_0^{2\pi} (\text{upper} - \text{lower})\,dx$$

$$= \int_0^{\pi/4} (\cos(x) - \sin(x))\,dx + \int_{\pi/4}^{5\pi/4} (\sin(x) - \cos(x))\,dx + \int_{5\pi/4}^{2\pi} (\cos(x) - \sin(x))\,dx$$

Two crossing points: where $\sin(x) = \cos(x)$ for $0 \le x \le 2\pi$

ie $x = \frac{\pi}{4}$ & $x = \frac{5\pi}{4}$

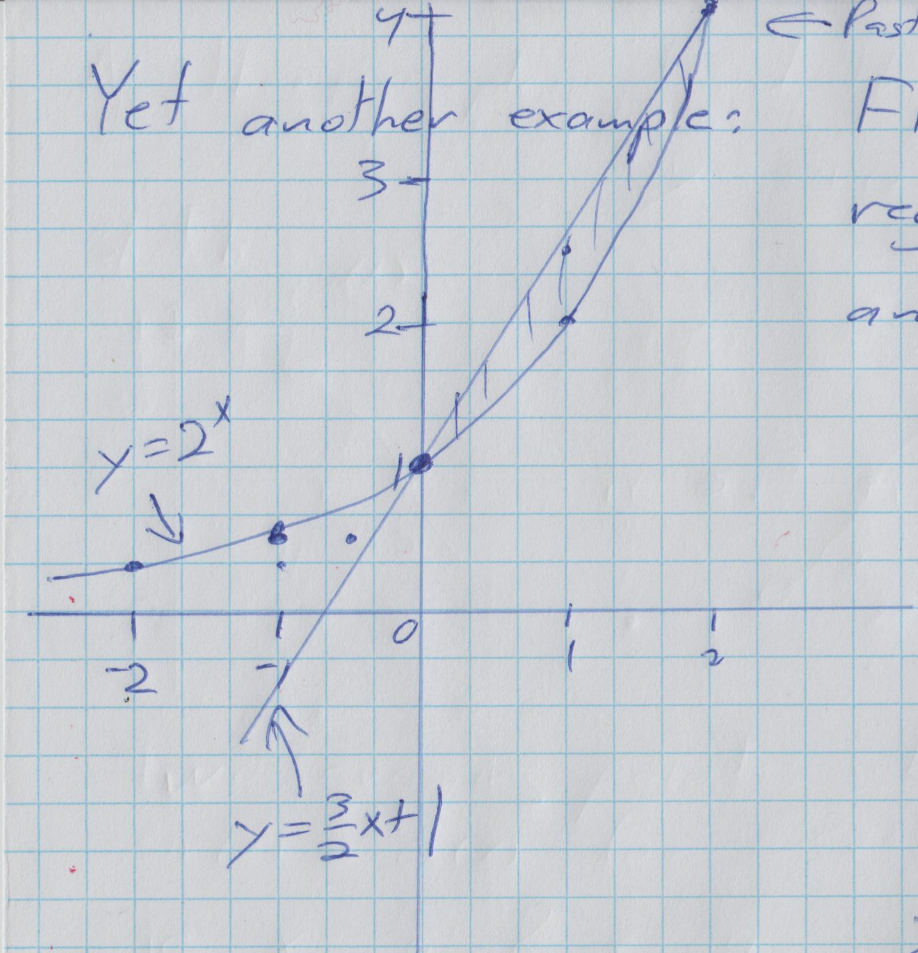
$$= \left[\sin(x) + (+\cos(x))\right]\Big|_0^{\pi/4} + \left[-\cos(x) - \sin(x)\right]\Big|_{\pi/4}^{5\pi/4} + \left[\sin(x) + (+\cos(x))\right]\Big|_{5\pi/4}^{2\pi}$$

$$= \left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1)\right] + \left[\left(+\frac{+1}{\sqrt{2}} + \frac{+1}{\sqrt{2}}\right) - \left(\frac{-1}{\sqrt{2}} + \frac{-1}{\sqrt{2}}\right)\right] + \left[(0+1) - \left(\frac{-1}{\sqrt{2}} + \frac{-1}{\sqrt{2}}\right)\right]$$

$$= \left[\frac{2}{\sqrt{2}} - 1\right] + \left[\frac{2}{\sqrt{2}} + \frac{+2}{\sqrt{2}}\right] + \left[1 - \frac{-2}{\sqrt{2}}\right]$$

$$= \sqrt{2} - \cancel{1} + 2\sqrt{2} + \cancel{1} + \sqrt{2} = 4\sqrt{2}$$

Yet another example: Find the area of the finite region bounded by $y = 2^x$ and $y = \frac{3}{2}x + 1$.



These curves intersect at $x=0$ & $x=2$. Before $x=0$ & after $x=2$ they never cross again, so the <u>finite</u> region they enclose is the one between 0 & 2, over which $y = \frac{3}{2}x + 1$ is above $y = 2^x$.

$$\text{Area} = \int_0^2 (\text{upper} - \text{lower})\,dx = \int_0^2 \Big(\underbrace{\tfrac{3}{2}x + 1}_{\text{upper}} - \underbrace{2^x}_{\text{lower}}\Big)\,dx$$

$$= \left(\frac{3}{2}\cdot\frac{x^2}{2} + x - \frac{2^x}{\ln(2)}\right)\Bigg|_0^2$$

$$= \left(\frac{3}{\cancel{4}}\cdot\cancel{2^2} + 2 - \frac{2^2}{\ln(2)}\right) - \left(\frac{3}{4}\cdot 0^2 - \frac{2^0}{\ln(2)}\right)$$

$$= 5 - \frac{4}{\ln(2)} + \left(+\frac{1}{\ln(2)}\right) = 5 - \frac{3}{\ln(2)}$$

$$\int 2^x\,dx = \int \left(e^{\ln(2)}\right)^x dx$$

$$= \int e^{\ln(2)\cdot x}\,dx \qquad u = \ln(2)\,x,\quad du = \ln(2)\,dx,\quad dx = \frac{1}{\ln(2)}\,du$$

$$= \int e^u \cdot \frac{1}{\ln(2)}\,du$$

$$= \frac{e^u}{\ln(2)} + C = \frac{e^{\ln(2)\cdot x}}{\ln(2)} + C = \frac{2^x}{\ln(2)} + C.$$