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Integration by Parts III – more examples, plus using parts to help simplify some trig integrals

Examples: 1° $\displaystyle\int_1^e x^3 \ln(x)\,dx$

Following the rule of thumb:

~~$u = x^3$ & $v' = \ln(x)$, so~~

~~$u' = 3x^2$ & $v =$~~

$u = \ln(x)$ & $v' = x^3$ so

$u' = \frac{1}{x}$ & $v = \frac{x^4}{4}$.

$$= \frac{x^4}{4}\ln(x)\Big|_1^e - \int_1^e \frac{1}{x}\cdot\frac{x^4}{4}\,dx$$

$$= \left(\frac{e^4}{4}\underbrace{\ln(e)}_{=1} - \frac{1^4}{4}\underbrace{\ln(1)}_{=0}\right) - \int_1^e \frac{x^3}{4}\,dx$$

$$= \left(\frac{e^4}{4} - 0\right) - \frac{1}{4}\int_1^e x^3\,dx = \frac{e^4}{4} - \frac{1}{4}\cdot\frac{x^4}{4}\Big|_1^e$$

$$= \frac{e^4}{4} - \left(\frac{e^4}{16} - \frac{1^4}{16}\right) = \frac{4e^4}{16} - \frac{e^4}{16} + \frac{1}{16} = \frac{3e^4}{16} - \frac{1}{16}$$

$$= \frac{1}{16}\left[3e^4 - 1\right]$$

2° $\int_1^4 x e^{\sqrt{x}}\,dx$ ②

Can't really apply the rule of thumb directly because $e^{\sqrt{x}}$ is awfully hard to find an antiderivative for. Simplify using a substitution...

$w = \sqrt{x}$

$\Rightarrow x = w^2$, so

$dx = 2w\,dw$

and

x	w
1	1
4	2

Now apply the rule of thumb:

$u = w^3 \quad v' = e^w$

$u' = 3w^2 \quad v = e^w$

$$= \int_1^2 w^2 e^w \cdot 2w\,dw = 2\int_1^2 w^3 e^w\,dw$$

$$= 2\left[w^3 e^w \Big|_1^2 - \int_1^2 3w^2 e^w\,dw \right]$$

$s = 3w^2 \quad t' = e^w$

$s' = 6w \quad t = e^w$

$$= 2\left[(2^3 e^2 - 1^3 e^1) - \left[3w^2 e^w \Big|_1^2 - \int_1^2 6we^w \right] \right]$$

$q = 6w \quad r' = e^w$

$q' = 6 \quad r = e^w$

$$= 2\left[(8e^2 - e) - (3\cdot 2^2\cdot e^2 - 3\cdot 1^2\cdot e^1) - \left[6we^w\Big|_1^2 - \int_1^2 6e^w\right]\right] \quad 3$$

$$= 2\left[8e^2 - e - (12e^2 - 3e) - \left[(6\cdot 2\cdot e^2 - 6\cdot 1\cdot e^1) - 6e^w\Big|_1^2\right]\right]$$

$$= 2\left[8e^2 - e - \left[12e^2 - 3e - \left[12e^2 - 6e - (6e^2 - 6e^1)\right]\right]\right]$$

$$= 2\left[8e^2 - e - \cancel{12e^2} + 3e + \cancel{12e^2} - \cancel{6e} - 6e^2 + \cancel{6e}\right]$$

$$= 2\left[2e^2 + 2e\right] = 4e(e+1)$$

Whew!

Next: apply parts to certain trig integrals ...

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3° a) $\int \cos^3(x)\,dx = \int \cos(x)\cos^2(x)\,dx$

(w/o parts)

$= \int \cos(x)(1-\sin^2(x))\,dx$

Substitute $u = \sin(x)$, so $du = \cos(x)\,dx$

$= \int (1-u^2)\,du$

$= u - \frac{u^3}{3} + C = \boxed{\sin(x) - \frac{\sin^3(x)}{3} + C}$

b) $\int \cos^3(x)\,dx = \int \cos(x)\cos^2(x)\,dx$

(with parts)

Split it up, so computing u' & v is as easy as possible.

$u = \cos^2(x)$ $\qquad v' = \cos(x)$

$u' = 2\cos(x)(-\sin(x))$ $\qquad v = \sin(x)$

$= \cos^2(x)\sin(x) - \int 2\cos(x)(-\sin(x))\sin(x)\,dx$

$= \cos^2(x)\sin(x) + 2\int \cos(x)\sin^2(x)\,dx$

Substitute $w = \sin(x)$ so $dw = \cos(x)\,dx$

$= \cos^2(x)\sin(x) + 2\int w^2\,dx$

$$= \cos^2(x)\sin(x) + 2\cdot\frac{w^3}{3} + C$$

$$= \cos^2(x)\sin(x) + \frac{2}{3}\sin^3(x) + C$$ Is this the same? Yes...

$$= \sin(x)\left[\cos^2(x) + \frac{2}{3}\sin^2(x)\right] + C$$

$$= \sin(x)\left[1 - \sin^2(x) + \frac{2}{3}\sin^2(x)\right] + C$$

$$= \sin(x)\left[1 - \frac{1}{3}\sin^2(x)\right] + C$$

$$= \sin(x) - \frac{1}{3}\sin^3(x) + C$$ Same as before...

Morals: ① Parts might not be needed, even if it's possible to use it.

② Trig integrals can have different-looking answers, that are actually the same.

4° $\boxed{\int \sin^n(x)\,dx}$ Use parts to reduce this to a lower power of sin in the integral.

$(n > 2)$

$u = \sin^{n-1}(x)$ & $v' = \sin(x)$,

so $u' = (n-1)\sin^{n-2}(x)\cdot\cos(x)$ & $v = -\cos(x)$

$$= -\cos(x)\sin^{n-1}(x) - \int (n-1)\sin^{n-2}(x)\cos(x)(-\cos(x))\,dx$$

$$= -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\cos^2(x)\,dx$$

$$= -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)(1-\sin^2(x))\,dx$$

$$= -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\,dx - (n-1)\boxed{\int \sin^n(x)\,dx}$$

so $n\boxed{\int \sin^n(x)\,dx} = -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\,dx$ & thus

Reduction formula for

$$\boxed{\int \sin^n(x)\,dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx}$$