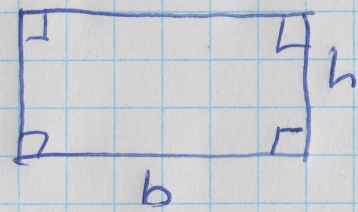
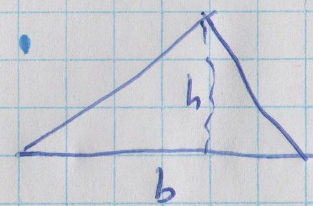
Finding Areas - welcome to integration and defining the definite integral

Easy for rectangles:



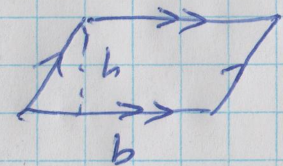
Area = bh

triangles:



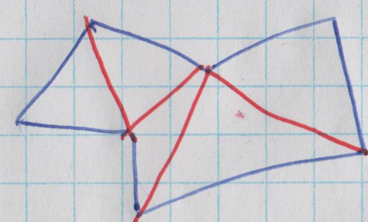
Area = $\frac{1}{2}$bh

parallelograms:



Area = bh

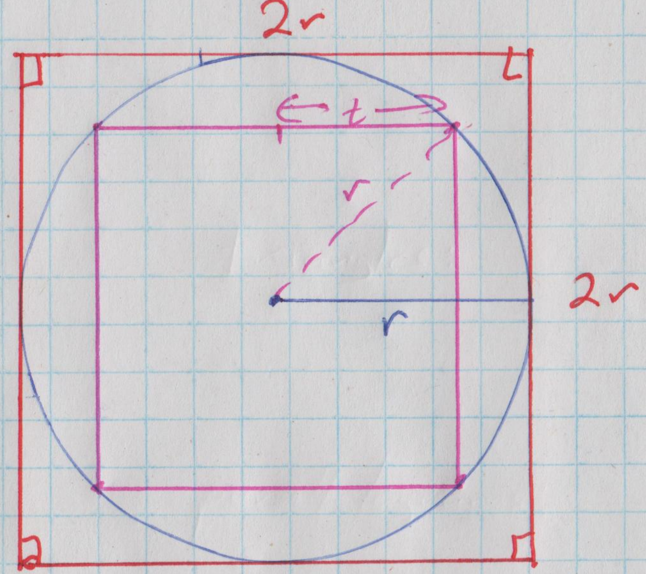
polygons:



Divide it up into triangles & add up their areas.

Harder for curved shapes: circles, parabolas, trig curves, ...

eg circle: We know the formula Area $= \pi \cdot \text{radius}^2$, but how do we get there from being able to do polygons?



circumscribed square is $2r \times 2r$

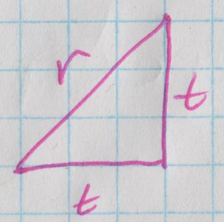
so Area of circle $< 4r^2$

inscribed square is $\sqrt{2}r \times \sqrt{2}r$

so Area of circle $> (\sqrt{2})^2 r^2 = 2r^2$

Thus $2 < \pi < 4$; using 96-gons Archimedes showed that

$$3.1408 \approx \underset{3\frac{10}{71}}{\frac{223}{71}} < \underset{3.1416}{\pi} < \underset{3\frac{1}{7}}{\frac{22}{7}} \approx 3.1429.$$

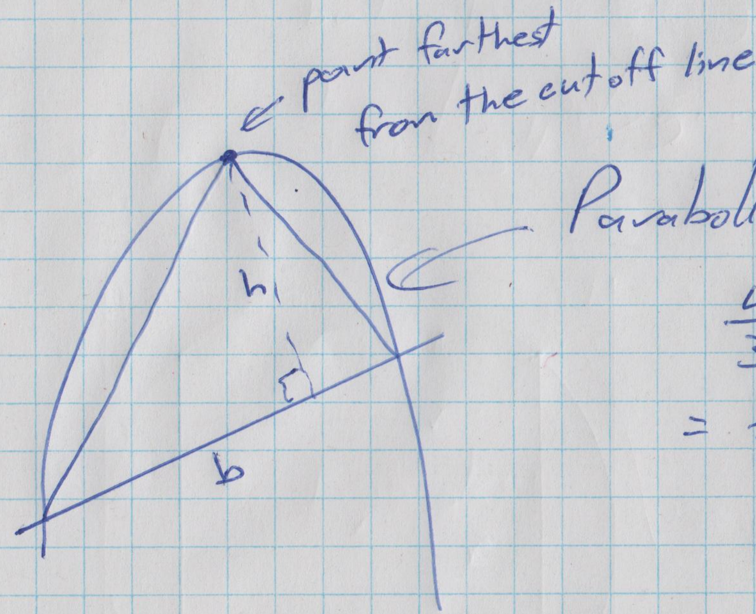


$r^2 = t^2 + t^2 = 2t^2$

$\Rightarrow r = \sqrt{2}\,t$

$\Rightarrow t = \frac{r}{\sqrt{2}}$ $\Rightarrow$ side of the inscribed square is $2t = \frac{2r}{\sqrt{2}} = \sqrt{2}\cdot r$

⇒ parabolas   Archimedes (287–212 BC) in Quadrature of the Parabola

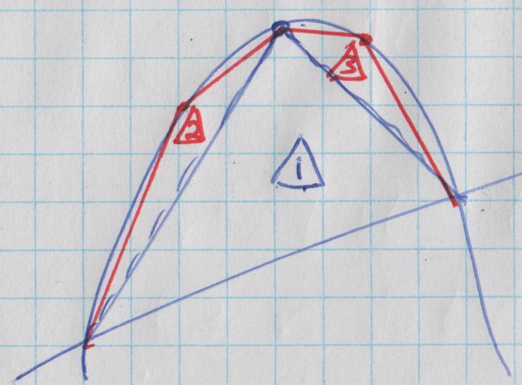


Parabolic segment has area

$\frac{4}{3}$ · area of the triangle

$= \frac{4}{3} \cdot \frac{1}{2} bh.$

How did he get this? The key result he needed is that if you repeat the process:



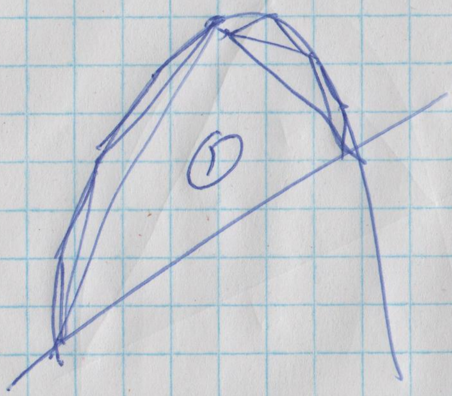
(area of △1) · $\frac{1}{4}$

= [(area of △2) + (area of △3)]

Archimedes does with Euclidean geometry & centers of mass of cross-sections...

If you repeat it again, you get 4 triangles whose total area is $\frac{1}{4}$ of .

Thus the area of the original parabolic segment



is (area of triangle 1)

$+ \frac{1}{4}($ — " — $)$

$+ \left(\frac{1}{4}\right)^2($ — " — $)$

$+ \left(\frac{1}{4}\right)^3($ — " — - - $)$

$\vdots$

$\frac{4}{3}$ (area of triangle 1)

||

$= \left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right)$ (area of triangle 1)

but

$$\left(1 - \frac{1}{4}\right)\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right) = 1 + \cancel{\tfrac{1}{4}} + \cancel{\tfrac{1}{4^2}} + \cancel{\tfrac{1}{4^3}} + \cancel{\cdots} - \cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{4^2}} - \cancel{\tfrac{1}{4^3}} + \cancel{\cdots} = 1$$

so $1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots = \frac{1}{1 - \frac{1}{4}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$

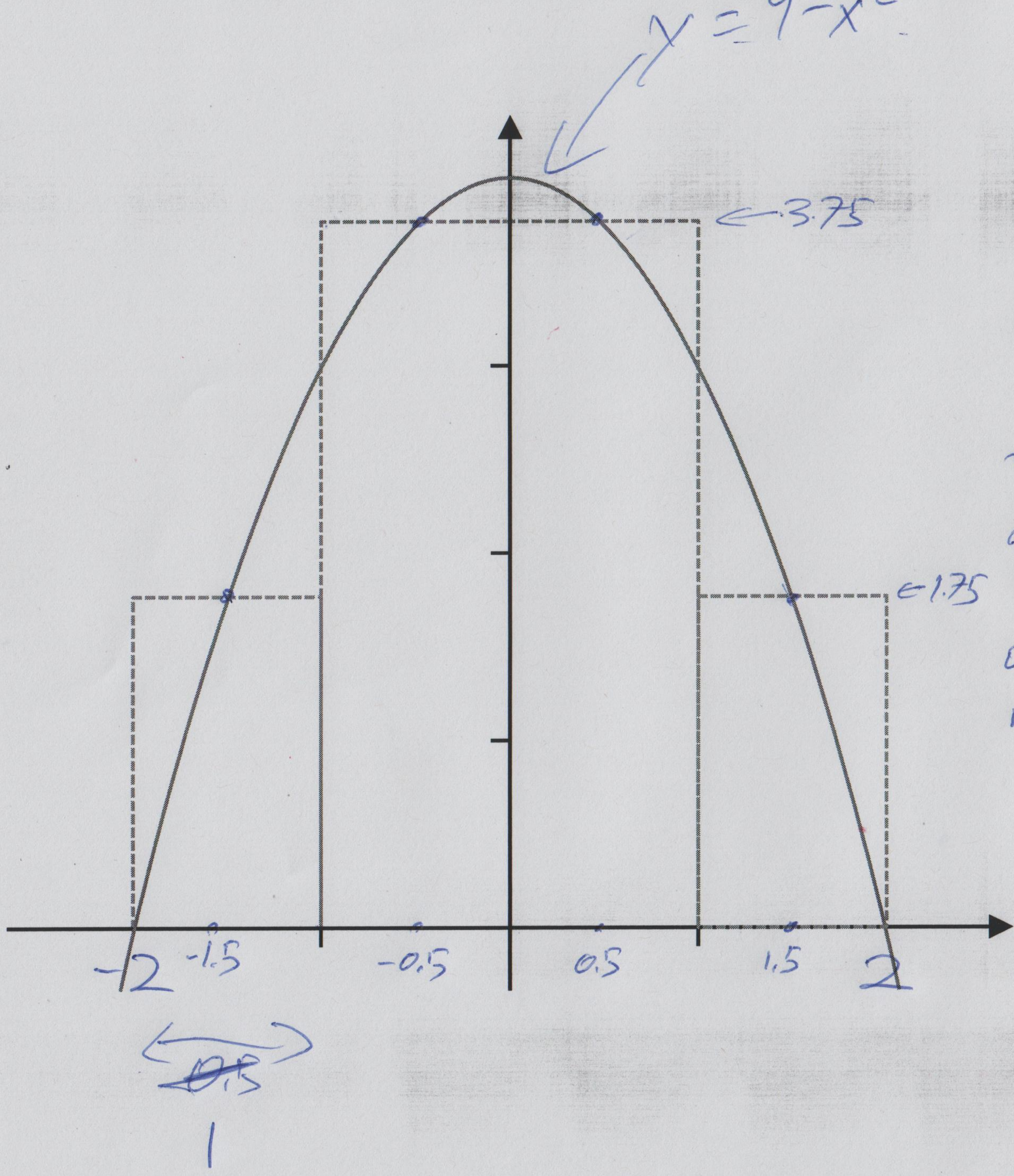
How do *we* do it? We approximate the region whose area we want by rectangles and then take a limit of the areas of the rectangles as you shrink their width to 0:

The area under $y = 4 - x^2$ & above the x-axis is?

- Approximating with four rectangles & heights determined at the midpoints of the base gives area 11 units^2.
- Approximating in the same way with eight rectangles gives area 10.75 units^2.

⋮

The limit turns out to be $10\frac{2}{3} = 10.666\ldots$ and thats the area under the parabola.

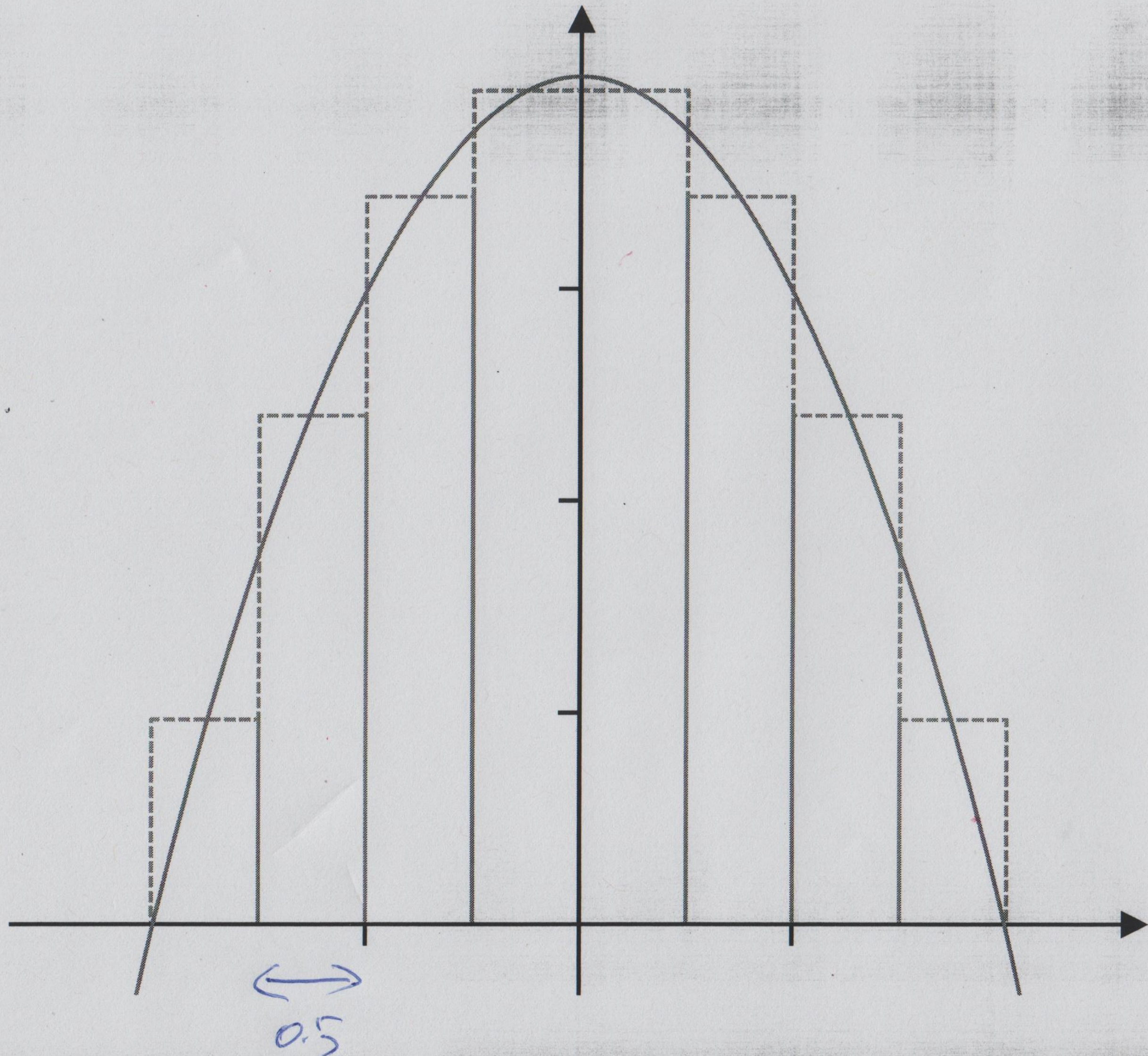


Area of the region between $y = 4 - x^2$ & the x-axis, approximated by 4 rectangles of equal width, namely $\frac{1}{2}$, and height determined by evaluating $4 - x^2$ at the midpoint of the base of each rectangle.

Area of rectangles

$$
\begin{aligned}
&= 1.75 \times \cancel{0.5}\ 1 \\
&+ 3.75 \times \cancel{0.5}\ 1 \\
&+ 3.75 \times \cancel{0.5}\ 1 \\
&+ 1.75 \times \cancel{0.5}\ 1 \\
&= 11
\end{aligned}
$$

5B



The calculation is twice as long... & works out to an area of 10.75 for all the rectangles...

This is a very cumbersome process, so we'll simplify it a bit by evaluating the heights at the right-hand endpoint of the base of each rectangle

For the region below $y = 4 - x^2$ and above the x-axis this gives an area (for n rectangles)

$$\sum_{i=1}^{n} \frac{4}{n}\left(4 - \left[\left(-2 + i\frac{4}{n}\right)^2\right]\right).$$

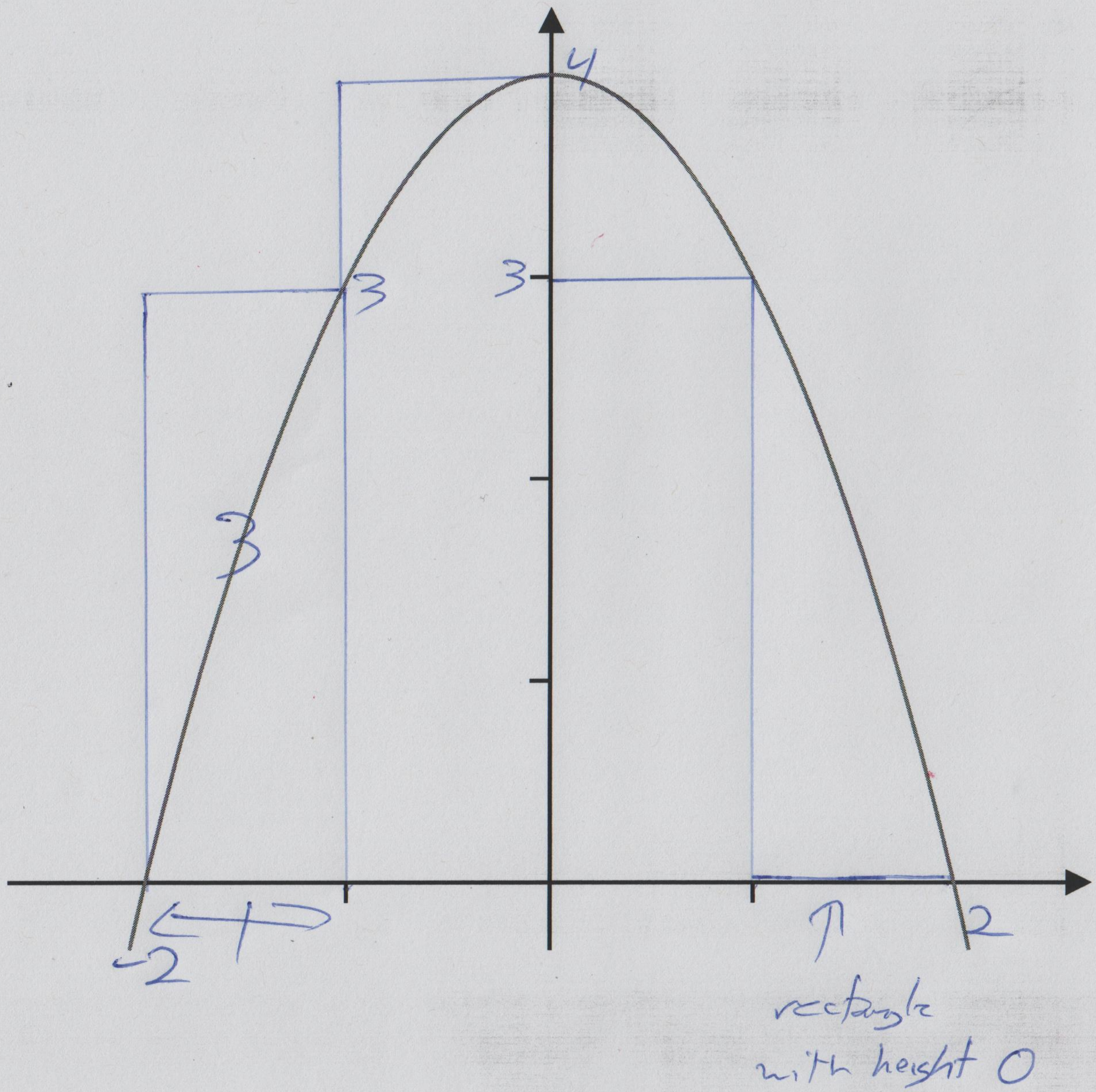
We can simplify and then take a limit of this:

$$= \sum_{i=1}^{n} \frac{4}{n}\left(4 - \left[(-2)^2 + 2(-2)i\frac{4}{n} + \left(i\frac{4}{n}\right)^2\right]\right)$$

$$= \sum_{i=1}^{n} \frac{4}{n}\left(\cancel{4} - \cancel{4} + \frac{16i}{n} - \frac{16i^2}{n^2}\right)$$

$$= \sum_{i=1}^{n} \frac{64}{n^2}\left(i - \frac{i^2}{n}\right) = \frac{64}{n^2}\sum_{i=1}^{n}\left(i - \frac{i^2}{n}\right)$$

6A

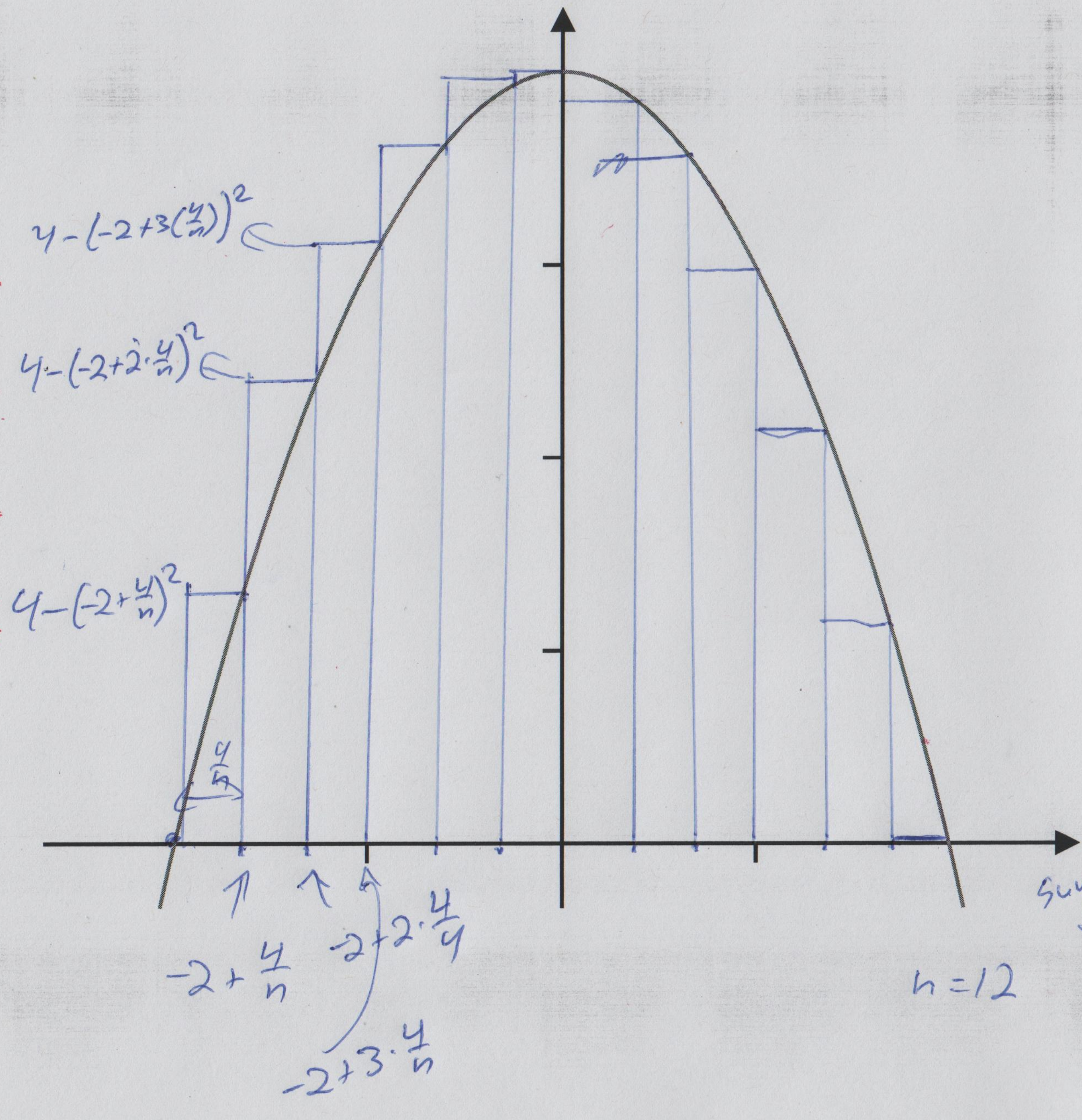


total area of the rectangles is

$3+4+3+0$

$= 10 \quad \left[< 10\tfrac{2}{3}\right]$

6B



If we divide up the interval $[-2,2]$ into n equal pieces. Each has width

$$\frac{2-(-2)}{n} = \frac{4}{n}$$

and areas

$$\frac{4}{n}\cdot\left(4-\left(-2+\frac{4}{n}\right)^2\right)$$

$$+\ \frac{4}{n}\cdot\left(4-\left[\cancel{2}\left(-2+2\frac{4}{n}\right)\right]^2\right)$$

$$+\ \frac{4}{n}\cdot\left(4-\left[\cancel{2}\left(-2+3\frac{4}{n}\right)\right]^2\right)$$

$$\vdots$$

$$+\ \frac{4}{n}\left(4-\left[\cancel{n}\left(-2+n\frac{4}{n}\right)\right]^2\right)$$

$$\|$$

sum

$$\sum_{i=1}^{n}\frac{4}{n}\left(4-\left[\cancel{4}\left(-2-i\frac{4}{n}\right)\right]^2\right)$$

$$= \frac{64}{n^2}\left[\left(\sum_{i=1}^{n} i\right) - \left(\sum_{i=1}^{n} \frac{i^2}{n}\right)\right]$$

$$= \left[\frac{64}{n^2}\cdot \sum_{i=1}^{n} i\right] - \left[\frac{64}{n^2}\cdot \sum_{i=1}^{n} \frac{i^2}{n}\right]$$

$$= \frac{64}{\cancel{n^2}}\cdot\frac{\cancel{n}(n+1)}{\cancel{2}} - \frac{64}{n^3}\cdot\left(\sum_{i=1}^{n} i^2\right)$$

$$= \frac{32}{n}(n+1) - \frac{64}{\cancel{n^3}\,n^2}\cdot\frac{\cancel{n}(n+1)(2n+1)}{6}$$

$$= 32\left(1+\frac{1}{n}\right) - \frac{32}{3n^2}\cdot\left(\frac{2n^2+3n+1}{1}\right)$$

$$= 32 + \underbrace{\frac{32}{n}}_{\downarrow\,0} - \frac{32}{3}\left(2 + \underbrace{\frac{3}{n}}_{\downarrow\,0} + \underbrace{\frac{1}{n^2}}_{\downarrow\,0}\right) \quad \text{as } n \to \infty$$

$$= 32 - \underbrace{\frac{32}{3}\cdot 2}_{} = \frac{32}{3}$$

$$\frac{96}{3} - \frac{64}{3}$$

Which is the actual area! Cumbersome process... and that's where calculus comes in!

$$\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$