# Trigonometric Integrals

(with much more than is in §8.2 of the text)

Want to handle integrals of the form

$$\int \sin^n(x)\cos^k(x)\,dx$$

or

$$\int \tan^n(x)\sec^k(x)\,dx\,.$$

Trig identities that might come up:

$\sin^2(x)+\cos^2(x)=1$ usually as $\sin^2(x)=1-\cos^2(x)$
or $\cos^2(x)=1-\sin^2(x)$

$\sec^2(x)=1+\tan^2(x)$ often as $\tan^2(x)=\sec^2(x)-1$

$\sin(2x)=2\sin(x)\cos(x)$

$$\begin{aligned}\cos(2x) &= \cos^2(x)-\sin^2(x)\\ &= 1-2\sin^2(x)\\ &= 2\cos^2(x)-1\end{aligned}$$

often as
$\cos^2(x)=\frac{1}{2}+\frac{1}{2}\cos(2x)$
$\sin^2(x)=\frac{1}{2}-\frac{1}{2}\cos(2x)$

$\frac{d}{dx}\sin(x)=\cos(x)$ $\quad\frac{d}{dx}\cos(x)=-\sin(x)$

$\frac{d}{dx}\tan(x)=\sec^2(x)$ $\quad\frac{d}{dx}\sec(x)=\sec(x)\tan(x)$

$\sin(-x)=-\sin(x)$ $\quad\cos(-x)=\cos(x)$

Using substitution & trig identities

$$\int \sin^2(x)\cos^3(x)\,dx$$

$$= \int \sin^2(x)\cos^2(x)\cos(x)\,dx$$

$$= \int \sin^2(x)\left(1-\sin^2(x)\right)\cos(x)\,dx \qquad \begin{array}{l} u = \sin(x) \\ du = \cos(x)dx \end{array}$$

$$= \int u^2(1-u^2)\,du$$

$$= \int (u^2-u^4)\,du = \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + C \quad \checkmark$$

$$\int \sin^2(x)\cos^2(x)\,dx \qquad \text{use } \sin(2x) = 2\sin(x)\cos(x)$$

$$= \int \left(\sin(x)\cos(x)\right)^2 dx$$

$$= \int \left(\frac{1}{2}\sin(2x)\right)^2 dx = \frac{1}{4}\int \sin^2(2x)\,dx$$

$$= \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos(4x)\right) dx \qquad \begin{array}{l} w = 4x \\ dw = 4dx \text{ so } \frac{1}{4}dw = dx \end{array}$$

$$= \frac{1}{4}\cdot\frac{1}{2}\int \left(1 - \cos(w)\right)\cdot\frac{1}{4}dw$$

$$= \frac{1}{32}\left(w - \sin(w)\right) + C = \frac{1}{32}\left(4x - \sin(4x)\right) + C$$

$$\int \sec^3(x)\tan(x)\,dx$$

$$= \int \sec^2(x)\sec(x)\tan(x)\,dx \qquad \begin{aligned} u &= \sec(x) \\ du &= \sec(x)\tan(x)\,dx \end{aligned}$$

$$= \int u^2\,du = \frac{u^3}{3} + C = \frac{1}{3}\sec^3(x) + C$$

$$\int \sec^3(x)\tan^2(x)\,dx = ? \qquad \text{use } \tan^2(x) = \sec^2(x) - 1$$

$$= \int \sec^3(x)\left(\sec^2(x) - 1\right) dx$$

$$= \int \left(\sec^5(x) - \sec^3(x)\right) dx$$

$$= \int \sec^5(x)\,dx - \int \sec^3(x)\,dx$$

Use integration by parts to develop a general "reduction" formula.

$$\int \sec^n(x)\,dx$$

Use parts

$u = \sec^{n-2}(x)$ $\quad v' = \sec^2(x)$

$u' = (n-2)\sec^{n-3}(x) \cdot \frac{d}{dx}\sec(x)$

$v = \tan(x)$

$= (n-2)\sec^{n-3}(x)\sec(x)\tan(x)$

$= (n-2)\sec^{n-2}(x)\tan(x)$

$$\boxed{\int \sec^n(x)\,dx} = \sec^{n-2}(x)\tan(x) - \int (n-2)\sec^{n-2}(x)\tan^2(x)\,dx$$

$$= \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)\left(\sec^2(x)-1\right)dx$$

$$= \sec^{n-2}(x)\tan(x) - (n-2)\int \left(\sec^n(x) - \sec^{n-2}(x)\right)dx$$

$$= \sec^{n-2}(x)\tan(x) - \boxed{(n-2)}\boxed{\int \sec^n(x)\,dx} + (n-2)\int \sec^{n-2}(x)\,dx$$

Solve for

$$\underbrace{\left(1+(n-2)\right)}_{n-1}\int \sec^n(x)\,dx = \sec^{n-2}(x)\tan(x) + (n-2)\int \sec^{n-2}(x)\,dx$$

$$\boxed{\int \sec^n(x)\,dx} = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

eg $\displaystyle\int \sec^3(x)\,dx = \frac{1}{3-1}\sec^{3-2}(x)\tan(x) + \frac{3-2}{3-1}\int \sec^{3-2}(x)\,dx$

$$= \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\int \sec(x)\,dx$$

So how do we integrate $\int \sec(x)\,dx$?

$$\int \sec(x)\,dx = \int \sec(x)\cdot 1\,dx$$

$$= \int \sec(x)\cdot \frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx$$

$$= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx \qquad \begin{aligned} u &= \sec(x)+\tan(x) \\ du &= (\sec^2(x)+\sec(x)\tan(x))\,dx \end{aligned}$$

$$= \int \frac{1}{u}\,du = \ln(u) + C$$

$$= \ln(\sec(x)+\tan(x)) + C$$

Similar formulas for reducing powers of $\cos(x)$ or $\sin(x)$ exist.