The basics of series (§11.2)

A series is an attempt to add up a sequence.

Given $\{a_n\}$, the series is $\sum_{n=0}^{\infty} a_n$

$$= a_0 + a_1 + a_2 + a_3 + a_4 + \cdots.$$

Warning: A series does not have to have a sensible sum eg $a_n = 1$ for all $n \geq 0$

$$\sum_{n=0}^{\infty} a_n = 1 + 1 + 1 + \cdots = \infty$$

& ∞ does not count as a real number.

Def'n: The partial sums of the series $\sum_{n=0}^{\infty} a_n$ are the finite sums

$$s_k = \sum_{n=0}^{k} a_n = a_0 + a_1 + \cdots + a_k.$$

The series $\sum_{n=0}^{\infty} a_n$ converges to a number $S \in \mathbb{R}$ ("adds up sensibly") if $\lim_{k \to \infty} s_k = S$.

Example: $a_n = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n \quad n \geq 0$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

Does this converge? If so, what to?

The partial sums of this series

$$s_k = \sum_{n=0}^{k} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k}$$

$$= \frac{1 - \frac{1}{2^{k+1}}}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{2^{k+1}}\right)$$

Then $\lim_{k\to\infty} s_k = \lim_{k\to\infty} 2\left(1 - \underbrace{\frac{1}{2^{k+1}}}_{\to 0}\right) = 2(1-0) = 2.$

Thus $\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges (to 2).

Terminology: A ~~seq~~ series that does not converge is said to diverge.

Example: Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

Does this converge? If so, what to?

Observe that the finite or partial sums of the harmonic series $s_k = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{k}$

have no obvious nice formula for the sum.

(No not so obvious one either...)

Suppose $k = 2^m$. Then s_k is

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{2}{4} = \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{4}{8} = \frac{1}{2}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{\geq \frac{8}{16} = \frac{1}{2}} +$$

$$\cancel{\geq} 1 \quad \frac{1}{2} \quad \cdots + \frac{1}{2^{m-1}} + \underbrace{\frac{1}{2^{m-1}+1} + \frac{1}{2^{m-1}+2} + \cdots + \frac{1}{2^m}}_{\geq \frac{2^{m-1}}{2^m} = \frac{1}{2}}$$

$$\geq 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{m \text{ copies}} = 1 + \frac{m}{2}$$

$$\lim_{k \to \infty} s_k = \lim_{m \to \infty} S_{2^m} \geq \lim_{m \to \infty} \left(1 + \frac{m}{2}\right) = \infty$$

$\therefore$ $\lim_{k \to \infty} s_k = \infty$ and <u>not</u> a real number,

so the harmonic series $\sum_{n=\phi}^{\infty} \frac{1}{n}$ <u>diverges</u>.

We'll develop a family of tests for whether a series converges or not.

Basic Properties of Series

1) If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ both converge, then so does $\sum_{n=0}^{\infty}(a_n+b_n)$ and

$$\sum_{n=0}^{\infty}(a_n+b_n) = \left(\sum_{n=0}^{\infty} a_n\right) + \left(\sum_{n=0}^{\infty} b_n\right).$$

2) If $\sum_{n=0}^{\infty} a_n$ converges and $c \in \mathbb{R}$, then so does $\sum_{n=0}^{\infty} ca_n$ and

$$\sum_{n=0}^{\infty} ca_n = c\left(\sum_{n=0}^{\infty} a_n\right).$$

3) If $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

4) [Divergence Theorem]

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ diverges.

Warning: Just because $\lim_{n\to\infty} a_n = 0$ does not mean that $\sum_{n=0}^{\infty} a_n$ converges. Example: $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges but $\lim_{n\to\infty} \frac{1}{n} = 0$.

Geometric Series

Given an <u>initial term</u> a (usually assume $a \neq 0$)
& a <u>common ratio</u> r,

$$\sum_{n=0}^{k} ar^n = a + ar + ar^2 + \cdots + ar^k = \frac{a(1-r^{k+1})}{1-r}$$

$$(1-r)(a + ar + ar^2 + \cdots + ar^k)$$

$$= a(1-r)(1 + r + r^2 + \cdots + r^k)$$

$$= a(1 + \cancel{r} + \cancel{r^2} + \cancel{\cdots} + \cancel{r^k} - \cancel{r} - \cancel{r^2} - \cancel{r^3} - \cdots - r^{k+1})$$

$$= a(1-r^{k+1})$$

The infinite series version is

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$$

<u>if</u> $|r|<1$

$$= \lim_{k\to\infty}\left(\sum_{n=0}^{k} ar^k\right) = \lim_{k\to\infty} a\frac{1-r^{k+1}}{1-r}$$

(otherwise it diverges)

$$r^{k+1} \to 0 \quad \text{if } |r|<1$$

but r^{k+1} has no limit (bounces or $\to\infty$) if $|r| \geq 1$.