Integrating Rational Functions II:

The method of partial fractions

We want to integrate rational functions

$$\int \frac{p(x)}{q(x)}\,dx \quad , \text{where } p(x) \,\&\, q(x) \text{ are polynomials.}$$

0. Factor out the leading coefficients in both polynomials and the resulting fraction out of the integral

(optional) but recommended

eg
$$\int \frac{3x^2+4x+9}{2x^3+3x^2-4x+16}\,dx$$

$$= \int \frac{3\left(x^2+\frac{4}{3}x+3\right)}{2\left(x^3+\frac{3}{2}x^2-2x+8\right)}\,dx$$

$$= \frac{3}{2}\int \frac{x^2+\frac{4}{3}x+3}{x^3+\frac{3}{2}x^2-2x+8}\,dx$$

1. Factor the denominator fully into (powers of) linear factors and (powers of) irreducible quadratic factors.

$$g(x) = (x^2+b_1x+c_1)^{k_1}\cdot(x^2+b_2x+c_2)^{k_2}\cdot\cdots(x^2+b_ix+c_i)^{k_i}$$
$$\cdot(x-a_1)^{l_1}\cdot(x-a_2)^{l_2}\cdot\cdots(x-a_j)^{l_j}$$

This is <u>hard</u> to do if the polynomial is of "high" degree.

We will usually have our denominator be really easy to factor or come pre-factored.

2. $\frac{p(x)}{g(x)} = \frac{p(x)}{\text{factored version}}$ can be written as "partial fractions"

$$= \frac{B_{1,1}x+C_{1,1}}{(x^2+b_1x+c_1)^1} + \frac{B_{1,2}x+C_{1,2}}{(x^2+b_1x+c_1)^2} + \cdots + \frac{B_{1,k_1}x+C_{1,k_1}}{(x^2+b_1x+c_1)^{k_1}}$$
$$+ \frac{B_{2,1}x+C_{2,1}}{(x^2+b_2x+c_2)^1} + \cdots + \frac{B_{2,k_1}x+C_{1,k_1}}{(x^2+b_2x+c_2)^{k_2}}$$
$$\vdots$$
$$+ \frac{A_{1,1}}{(x-a_1)^1} + \frac{A_{1,2}}{(x-a_1)^2} + \cdots + \frac{A_{1,l_1}}{(x-a_1)^{l_1}}$$
$$+ \frac{A_{2,1}}{(x-a_2)^1} + \frac{A_{2,2}}{(x-a_2)^2} + \cdots + \frac{A_{2,l_2}}{(x-a_2)^{l_2}}$$
$$\vdots$$

$$\Rightarrow \frac{3x^2+2x+1}{(x^2+4)(x^2-2x+1)}$$

Since $(x^2+4)(x^2-2x+1) = (x^2+4)^1(x-1)^2$

$$= \frac{Bx+C}{x^2+4} + \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2}$$

3. Put the partial decomposition over a common denominator, rationalize the numerator and set it equal to the original numerator. This gives you a system of linear equations that let you solve for the unknown constants in the partial fraction decomposition.

$$\frac{3x^2+2x+1}{(x^2+4)(x-1)^2} = \frac{Bx+C}{x^2+4} + \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2}$$

$$= \frac{(Bx+C)(x-1)^2 + A_1(x^2+4)(x-1) + A_2(x^2+4)}{(x^2+4)(x-1)^2}$$

$$= \frac{(Bx+C)(x^2-2x+1) + A_1(x^3-x^2+4x-4) + A_2x^2+4A_2}{(x^2+4)(x-1)^2}$$

$$= \frac{Bx^3-2Bx^2+Bx+Cx^2-2Cx+C+A_1x^3-A_1x^2+4A_1x-4A_1+A_2x^2+4A_2}{(x^2+4)(x-1)^2}$$

$$= \frac{(B+A_1)x^3+(-2B+C-A_1+A_2)x^2+(B-2C+4A_1)x+(C-4A_1+4A_2)}{(x^2+4)(x-1)^2}$$

For the numerators to be equal the coefficients of each power of x must be equal.

$$\boxed{\begin{array}{ll} B+A_1=0 & -2B+C-A_1+A_2=3 \\ B-2C+4A_1=2 & C-4A_1+4A_2=1 \end{array}}$$

4. Solve the system of linear equations for the unknown constants in the partial fraction expansion

$B + A_1 = 0$ $\Rightarrow$ $A_1 = -B$ so

$B - 2C + 4A_1 = 2$

$-2B + C - A_1 + A_2 = 3$

$C - 4A_1 + 4A_2 = 1$

$$\boxed{\begin{aligned} -3B - 2C &= 2 \\ -B + C + A_2 &= 3 \\ C + 4B + 4A_2 &= 1 \end{aligned}}$$

$\Downarrow$

$C = 3 + B - A_2$ so

$-3B - 2(3 + B - A_2) = 2$

$\Rightarrow \boxed{-5B + 2A_2 = 8}$

$(3 + B - A_2) + 4B + 4A_2 = 1$

$\boxed{5B + 3A_2 = -2}$

$\Rightarrow 5B = -2 - 3A_2$

$\Rightarrow -(-2 - 3A_2) + 2A_2 = 8$

$\Rightarrow 5A_2 = 8 - 2 = 6$

$\Rightarrow \boxed{A_2 = \frac{6}{5}} = 1.2$

So $5B + 3 \cdot \frac{6}{5} = -2$

$\Rightarrow 5B = -2 - \frac{18}{5} = -\frac{28}{5}$

$\boxed{B = -\frac{28}{25}}$

so

$C = 3 + \frac{-28}{25} - \frac{6}{5}$

$= \frac{75 - 28 - 30}{25}$

$\boxed{C = \frac{17}{25}}$

$\boxed{A_1 = -\left(-\frac{28}{25}\right) = \frac{28}{25}}$

Thus

$$\frac{3x^2+2x+1}{(x^2+4)(x-1)^2} = \frac{Bx+C}{x^2+4} + \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2}$$

$$= \frac{-\frac{28}{25}x + \frac{17}{25}}{x^2+4} + \frac{\frac{28}{25}}{x-1} + \frac{\frac{6}{5}}{(x-1)^2}$$

~~5~~. Integrate the partial fraction expansion. ie

$$\int \frac{3x^2+2x+1}{(x^2+4)(x-1)^2}\,dx = \int \frac{-\frac{28}{25}x + \frac{17}{25}}{x^2+4}\,dx$$

$$+ \int \frac{\frac{28}{25}}{x-1}\,dx$$

$$+ \int \frac{6/5}{(x-1)^2}\,dx$$

substitute for $x-1$

$$\int \frac{-\frac{28}{25}x + \frac{17}{25}}{x^2+4} = \int \frac{-\frac{28}{25}x}{x^2+4}\,dx + \int \frac{\frac{17}{25}}{x^2+4}$$

substitute $w = x^2+4$

substitute $x = 2\tan(t)$

To simplify irreducible quadratics use combinations of completing the square and substitution.

$$\int \frac{1}{(u^2+1)^k}\,du = \frac{1}{2k-2}\cdot\frac{x}{(u^2+1)^{k-1}} + \frac{2k-3}{2k-2}\int\frac{1}{(u^2+1)^{k-1}}\,dx$$

<u>Warning</u> Partial fraction decomposition requires the degree of the denominator to be greater than the degree of the numerator. If that's the case do division to get around this problem before you do anything else.