Integrating Rational Functions I:
The algebraic preliminaries

A rational function is a ratio of polynomials eg $\dfrac{3x^2+4x+19}{4x^3-3x+21}$.

1. Polynomials can be divided with the caveat that divisor must have degree no greater than the degree of the one divided into.

Basic method is long division

Say we want to divide x^2+3x-1 into $2x^4+3x^3-x-13$.

$$
\begin{array}{r}
2x^2 - 3x + 11 \\
x^2+3x-1 \overline{\big)\, 2x^4 + 3x^3 + 0x^2 - x - 13} \\
-(2x^4 + 6x^3 - 2x^2) \\
\hline
-3x^3 + 2x^2 - x \\
-(-3x^3 - 9x^2 + 3x) \\
\hline
11x^2 - 4x - 13 \\
-(11x^2 + 33x - 11) \\
\hline
-37x - 2
\end{array}
$$

remainder term

so $2x^4 + 3x^3 - x - 13$

$= (x^2+3x-1)(2x^2-3x+11) - 37x - 2$

Guaranteed: the remainder has lower degree than the divisor

Application we'll be using is

$$\frac{2x^4+3x^3-x-13}{x^2+3x-1} = \frac{(x^2+3x-1)(2x^2-3x+11)-37x-2}{x^2+3x-1}$$

$$= 2x^2 - 3x + 11 + \frac{-37x-2}{x^2+3x-1}$$

degree of numerator < degree of denominator

2. <u>Fact:</u> Any polynomial can be factored into a product of linear terms, <u>eg</u> $x-a$, and irreducible quadratic terms

<u>eg</u> ax^2+bx+c where there are no roots

Quadratic formula tells you that ax^2+bx+c has roots

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

(if $b^2-4ac<0$, then there are no real roots so the quadratic is irreducible)

<u>Note:</u> if $p(a)=0$ for a polynomial $p(x)$ & number a, then $x-a$ is a factor of $p(x)$.

Factoring polynomials is hard:

Quadratic formula for degree 2.

Cubic & quartic formulas for degree 3 & 4.

<u>No general formula for degree 5 or more.</u>

Mostly we'll have denominators come at least partly pre-factored.

3. Complete the square in an (irreducible) quadratic

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$

$$\left(x+\frac{b}{2a}\right)^2 = x^2+2\frac{b}{2a}x+\frac{b^2}{4a^2}$$

$$= a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b}{4a^2}+\frac{c}{a}\right)$$

This preps for a substitution

$$u = x+\frac{b}{2a}$$

that simplifies the expression to $a(u^2 \pm \text{constant})$.

4. Solving systems of linear equations ~~for~~ (using back-substitution)

Suppose
$$\begin{aligned} A-B+C &= 3 \\ A+2B &= 4 \\ A+B-3C &= 5 \end{aligned}$$

Isolate one of the unknown, substitute into the otherequations, repeat as necessary until we have unknown = number.

$$A + 2B = 4 \Rightarrow A = 4 - 2B$$

Replace A by $4-2B$ in the other equations:

$$A - B + C = 3 \Rightarrow \boxed{-3B + C = -1}$$

$$(4-2B) - B + C$$

$$A + B - 3C = 5 \Rightarrow \boxed{-B - 3C = 1}$$

$$(4-2B) + B - 3C$$

$$-3B + C = -1 \Rightarrow C = 3B - 1$$

Replace C by $3B-1$ in the other equation

$$-B - 3(3B-1) = 1 \Rightarrow -10B = -2$$

$$-B - 9B + 3 \qquad \Rightarrow B = \frac{-2}{-10} = \frac{1}{5}$$

Substitute back to get the other unknowns

$$C = 3B - 1 = 3 \cdot \frac{1}{5} - 1 = -\frac{2}{5}$$

$$\& \quad A = 4 - 2B = 4 - 2 \cdot \frac{1}{5} = \frac{18}{5}$$

<u>Solution:</u> $A = \frac{18}{5} = 3\frac{3}{5}$, $B = \frac{1}{5}$, $C = -\frac{2}{5}$