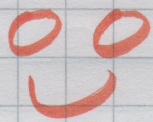


Just for fun! Not on quiz ^①
or exam!



Why the standard normal density is a valid density,

or, Behold the power of fully operational multivariate calculus! (With apologies to former Senator Palpatine...)

A function $f(x)$ is a valid probability density function if

(1) $f(x)$ is defined and integrable on \mathbb{R} ,

(2) $f(x) \geq 0$ for all $x \in \mathbb{R}$,

and (3) $\int_{-\infty}^{\infty} f(x) dx = 1$.

The standard normal density is

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Conditions (1) & (2) are easy to check for $g(x)$. (3) will take a bit of work...

To Show: $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1$

②

We'll actually show (and it's good enough)

that $\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right)^2 = 1.$

$$\left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right]^2$$

$$= \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right] \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right]$$

$$= \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right] \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy \right] dy$$

$$= \left(\frac{1}{\sqrt{2\pi}} \right)^2 \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-x^2/2} e^{-y^2/2} dy \right] dy$$

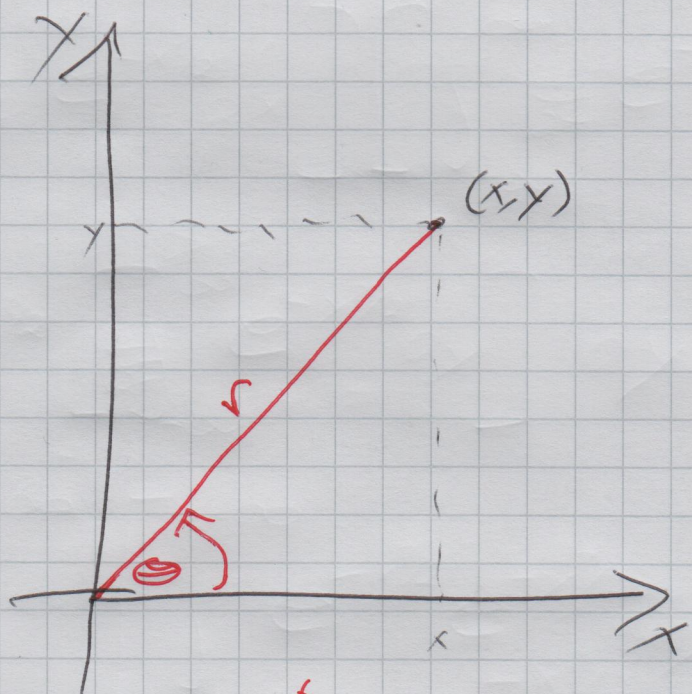
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dy dx$$

$$\frac{dA}{dx}$$

3

$$= \frac{1}{2\pi} \iint_{\mathbb{R}^2} e^{-(x^2+y^2)/2} dA$$

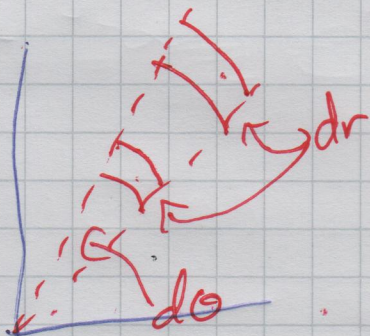
We're going to switch this to (& evaluate in) polar coordinates:



$$r = \sqrt{x^2 + y^2}$$

$$x = r \cos(\theta)$$

$$y = r \sin(\theta)$$

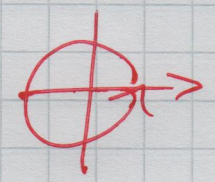


$$dA = r dr d\theta$$

9

$$= \frac{1}{2\pi} \iint_{\mathbb{R}^2} e^{-(x^2+y^2)/2} dA$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2} r dr d\theta$$



$0 \leq \theta \leq 2\pi$
radians

For the inner integral,

we'll use the substitution $u = -r^2/2$

r	u	
0	0	$= -0^2/2$
∞	$-\infty$	$= -\infty^2/2$

$$du = -\frac{dr}{\cancel{2}} dr$$

$$(-1)du = r dr$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{-\infty} e^u (-1) du d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_{-\infty}^0 e^u du d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (e^u \Big|_{-\infty}^0) d\theta = \frac{1}{2\pi} \int_0^{2\pi} (e^0 - e^{-\infty}) d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} 1 d\theta = \frac{1}{2\pi} \theta \Big|_0^{2\pi} = \frac{1}{2\pi} \cdot 2\pi - \frac{1}{2\pi} \cdot 0 = 1$$

