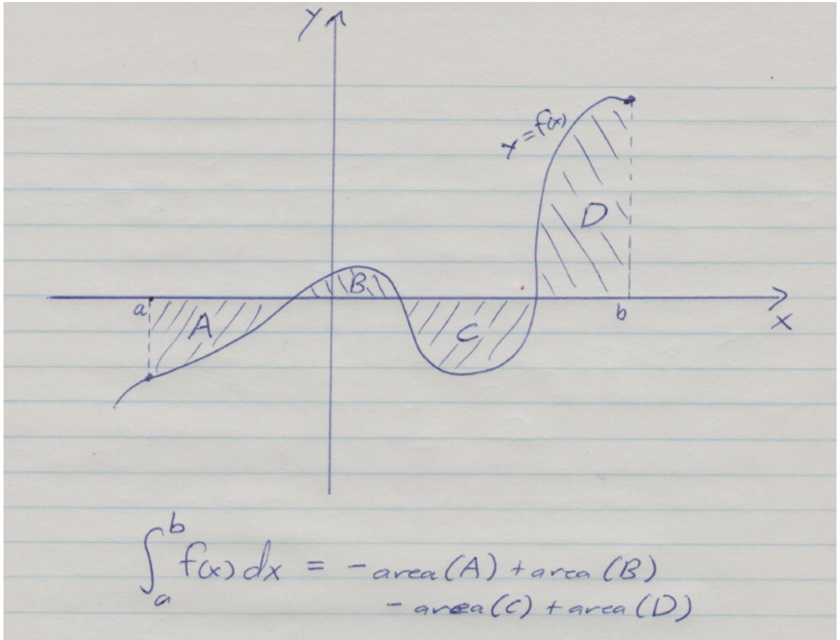
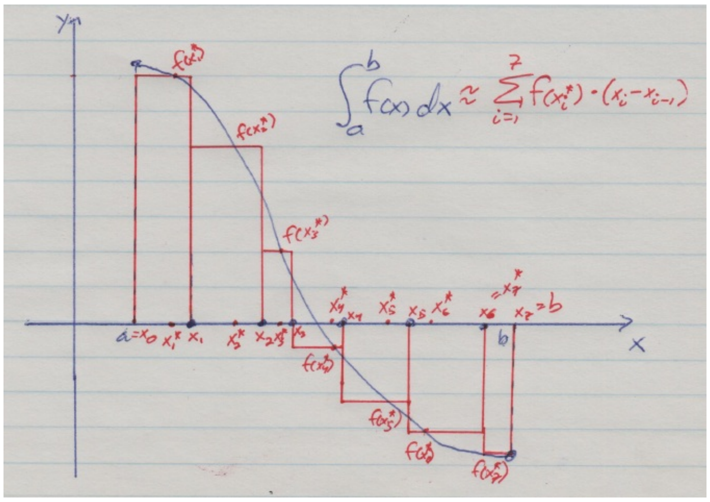
Informal Integrals

The Definite Integral

Informally, the definite integral $\int_a^b f(x)\,dx$ represents a weighted area between $y = f(x)$ and the x-axis for $a \le x \le b$, with area below the x-axis being subtracted and area above the x-axis being added.



There are several ways to define the definite integral formally. The first rigorous definition was due to Bernhard Riemann (1826-1866). His basic idea was to approximate the area between $y = f(x)$ and $y = 0$ by rectangles.



One subdivides the interval $[a, b]$ into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, … , $[x_{n-1}, x_n]$, where $x_0 = a$ and $x_n = b$. ($n = 7$ in the figure above.) Then one picks an x_i^* in $[x_{i-1}, x_i]$ for each i from 1 to n. The ith rectangle then has base $[x_{i-1}, x_i]$, and hence width $x_i - x_{i-1}$, and height $f(x_i^*)$, for an area of $f(x_i^*) \cdot (x_i - x_{i-1})$. Note that if $f(x_i^*)$ is negative, the rectangle's area is also negative. The sum of the areas of these rectangles approximates the weighted area between $y = f(x)$, that is:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^*) \cdot (x_i - x_{i-1})$$

As one makes the rectangles narrower and increases their number, one can get better approximations. Taking a suitable limit lets one use this idea to define the definite integral. Unfortunately, doing all of this both precisely and flexibly enough to get the desired basic properties of the definite integral out of the definition is surprisingly hard. (Take a peek at the handout *A Precise Definition of the Definite Integral* for the neatest version of Riemann integration known to your instructor. It's due to Jean-Gaston Darboux (1842-1917) and is well within reach of a first-year calculus student willing to put in a fair bit of work, but it would take an unacceptable amount of time to work through.) A little later, we will take a peek at one simplification of the Riemann integral which is a useful start on numerically approximating definite integrals, even though it is not flexible enough to serve as definition.

The Right-Hand Rule

The Right-Hand Rule approximation to the definite integral is obtained by taking the basic idea behind Riemann sums and simplifying it by having all the subintervals have equal width and by getting the height of each rectangle by evaluating the function at the right-hand endpoint of its base subinterval.

Suppose one subdivides the interval $[a, b]$ into n equal subintervals. Each subinterval will then have width $\frac{b-a}{n}$, so the first interval will be $\left[a, a + \frac{b-a}{n}\right]$, the second will be $\left[a + \frac{b-a}{n}, a + 2\frac{b-a}{n}\right]$, the third will be $\left[a + 2\frac{b-a}{n}, a + 3\frac{b-a}{n}\right]$, and so until the last subinterval, namely $\left[a + (n-1)\frac{b-a}{n}, a + n\frac{b-a}{n}\right] = \left[a + \frac{b-a}{n}, b\right]$. Note that the ith subinterval (for each of $i = 1$ up to n) is $\left[a + (i-1)\frac{b-a}{n}, a + i\frac{b-a}{n}\right]$ and its right-hand endpoint is $x_i^* = a + i\frac{b-a}{n}$.

Since $x_i - x_{i-1} = \frac{b-a}{n}$ for each subinterval, plugging all this into the basic scheme for a Riemann sum yields the following:

$$\sum_{i=1}^{n} f(x_i^*) \cdot (x_i - x_{i-1}) = \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right) \cdot \frac{b-a}{n} = \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)$$

The sum on the right is the Right-Hand Rule approximation to the definite integral, *i.e.* $\displaystyle\int_a^b f(x)\,dx \approx \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)$. In principle, one can use this formula to compute the definite integral exactly by taking a limit as $n \to \infty$,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right),$$

but this isn't practical in most cases, the major exception being low-degree polynomials.

While conceptually simple and somewhat useful, the Right-Hand Rule and its self-explanatory relatives the Left-Hand and Midpoint Rules aren't usually the best way to compute numerical approximations either, mainly because you usually need large ns, and hence a lot of arithmetic, to get good accuracy. More sophisticated approximations which still use equal subintervals such as the Trapezoid Rule and Simpson's Rule (see §8.6 in the textbook) do a better of getting precision with less overall arithmetic. Really sophisticated algorithms adapt the width of the subintervals to the slope of the function (the steeper the graph, the narrower you make your rectangles) or discard the idea of Riemann sums entirely and use very different ways to compute weighted areas.

Basic Properties of the Definite Integral

Assuming we've successfully defined it in some way, the definite integral has several properties related to the interval over which one is integrating:

1. Reversing the direction of the integral changes its sign: $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.
2. The area between $y = f(x)$ and the x-axis at a single point is 0: $\displaystyle\int_a^a f(x)\,dx = 0$.
3. The integral over two adjacent intervals is the integral over the combined interval:
$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

It has some related to the function being integrated (the *integrand*):

4. Scaling the function by a constant scales the integral by the same constant:
$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$
5. Integrating the sum of two functions is the same as summing the individual integrals of the functions: $\displaystyle\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

The above two properties mean that computing definite integrals is a *linear* process in the sense of linear algebra, since it respects addition and scaling by constants. This is especially useful in decomposing many integrals into ones that are simpler to compute.

The definite integral also has some occasionally useful order properties related to the usual order on the real numbers:

6. If $m \le f(x) \le M$ for all x in $[a, b]$ for some constants m and M, then
$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$
7. More generally, if $f(x) \le g(x)$ for all x in $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.
8. For any function $f(x)$ and interval $[a, b]$, $\displaystyle\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.

9. If a function $f(x)$ is continuous on $[a, b]$, then $\displaystyle\int_a^b |f(x)|\, dx = 0$ can only happen if $f(x) = 0$ for all x in $[a, b]$.
10. More generally, if $f(x)$ and $g(x)$ are any functions which are both continuous on $[a, b]$, then $\displaystyle\int_a^b |f(x) - g(x)|\, dx = 0$ can only happen if $f(x) = g(x)$ for all x in $[a, b]$.

The Fundamental Theorem of ~~Claculus~~* Calculus

Miraculously, there is a connection between definite integrals and derivatives which makes many definite integrals possible to compute precisely:

i. If $F(x)$ is a differentiable function on $[a, b]$ and $f(x) = F'(x)$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)\,.$$

ii. Assuming that we can integrate a function $f(t)$, if we define a new function $F(x)$ by $F(x) = \displaystyle\int_a^x f(t)\, dt$, then $F(x)$ will be differentiable and $F'(x) = f(x)$.

There are also some technical conditions and limitations I've omitted above, but you're not likely to run into one of the functions they deal with in this course and quite likely anywhere else. (Well, unless you take a lot more math ... :-) Making the proofs of the Fundamental Theorem of Calculus work is one reason defining the definite integral precisely is both important and treacherous.

What the first form of theorem does for us is let us the reduce the problem of computing a definite integral $\displaystyle\int_a^b f(x)\, dx$ to finding an anti-derivative for the integrand $f(x)$, *i.e.* a function $F(x)$ such that $F'(x) = f(x)$, and then evaluating said anti-derivative. The fact that the derivatives of most basic functions we know are (variations of) other basic functions lets us quickly start building a ...

A Library of Basic Anti-Derivatives

If we run each derivative we know in reverse, we get a corresponding antiderivative relation. Note that because the derivative of a constant is 0, two functions that differ by a constant will have the same derivative. What this means for us is that when are just considering anti-derivatives for the purposes of integration, *i.e. indefinite integrals*, we have to put in a generic "constant of integration" to stay honest: if $F(x)$ is a particular anti-derivative of $f(x)$, we write the indefinite integral as $\int f(x)\, dx = F(x) + C$. This

* If we only changed the name of the subject a little, we could call ourselves "clackers" ... Apologies to the authors of *The Difference Engine*, William Gibson and Bruce Sterling, an alternate history novel in which Charles Babbage succeded in building the mechanical computer he planned. The programmers of these "difference engines" were called clackers because of the noise the machines made.

constant doesn't matter for the purpose of computing definite integrals because it cancels out:

$$\int_a^b f(x)\,dx = (F(b)+C) - (F(a)+C) = F(b)+C-F(a)-C = F(b)-F(a)$$

However, the constant does matter in other applications of anti-derivatives, such as solving differential equations. In such applications you usually use some of the given information to actually find the value of the constant for that particular problem. For example, to solve the differential equation

$$\frac{dy}{dx} = 2x, \text{ with initial condition } y(1) = 2,$$

we find the generic antiderivative of $2x$, namely $y(x) = x^2 + C$, and then solve for C by using the initial condition: $2 = y(1) = 1^2 + C$, so $C = 2 - 1^2 = 1$. Thus the solution to the given problem is the particular function $y(x) = x^2 + 1$.

Because of these other applications we will always include the generic constant when writing out indefinite integrals; because these constants don't matter when using indefinite integrals to compute definite integrals, we will simply ignore them when computing definite integrals.

With all that out of the way, here is a start on our library of basic antiderivatives:

Derivative formula	*Indefinite integral*	*Conditions*
$\frac{d}{dx}x^k = kx^{k-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$	if $n \neq -1$
$\frac{d}{dx}\ln(x) = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln(x) + C$	
$\frac{d}{dx}e^x = e^x$	$\int e^x\,dx = e^x + C$	
$\frac{d}{dx}\sin(x) = \cos(x)$	$\int \cos(x)\,dx = \sin(x) + C$	
$\frac{d}{dx}\cos(x) = -\sin(x)$	$\int \sin(x)\,dx = -\cos(x) + C$	*(because?)*
$\frac{d}{dx}\arctan(x) = \frac{1}{x^2+1}$	$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$	

Starting with these and using the various techniques of integration we are going to study we will be able to find the antiderivatives, and hence compute the definite integrals, of a pretty wide range of functions. The first rule we'll state is, in fact, encapsulated in the first two indefinite integrals on our list above. The *Power Rule for Integration* is:

$$\int x^n\,dx = \begin{cases} \dfrac{x^{n+1}}{n+1} + C & \text{if } n \neq -1 \\ \ln(x) + C & \text{if } n = -1 \end{cases}$$

Between the Power Rule and the basic properties of definite integrals we can already integrate all of one major class of functions, namely polynomials. For example:

$$\begin{aligned}
\int_1^2 \left(x^2+3x-4\right)\,dx &= \int_1^2 x^2\,dx + \int_1^2 3x\,dx - \int_1^2 4\,dx \\
&= \left.\frac{x^3}{3}\right|_1^2 + 3\int_1^2 x\,dx - 4\int_1^2 1\,dx \\
&= \left(\frac{2^3}{3}-\frac{1^3}{3}\right) + 3\left.\frac{x^2}{2}\right|_1^2 - 4x|_1^2 \\
&= \left(\frac{8}{3}-\frac{1}{3}\right) + \left(3\frac{2^2}{2}-3\frac{1^2}{2}\right) - (4\cdot 2 - 4\cdot 1) \\
&= \frac{7}{3} + \left(6-\frac{3}{2}\right) - (8-4) \\
&= \frac{7}{3}+\frac{9}{2}-4 = \frac{14}{6}+\frac{27}{6}-\frac{24}{6} = \frac{17}{6}
\end{aligned}$$

Try to go through this and see where each basic property and the Power Rule were used and how.

Next: the (Basic) Substitution Rule.