Improper Integrals

(In Section 9.7 of the textbook.)

or,

How to deal with asymptotes when integrating.

Need this for handling infinite series.

Improper integrals with an ∞ or $-\infty$ in the limits.

"∞" is not a real number...

eg $\displaystyle\int_0^\infty x e^{-x}\,dx$

We need limits. (Review!)

$$= \lim_{t\to\infty} \int_0^t x e^{-x}\,dx$$

$u = x \quad v' = e^{-x}$

$u' = 1 \quad v = -e^{-x}$

$$= \lim_{t\to\infty} \left[-xe^{-x}\Big|_0^t - \int_0^t 1\cdot(-e^{-x})\,dx \right]$$

$$= \lim_{t\to\infty} \left[(-te^{-t}) - \cancel{(-0 e^{-0})} + \int_0^t e^{-x}\,dx \right]$$

$$= \lim_{t\to\infty} \left[-te^{-t} + (-e^{-x})\Big|_0^t \right]$$

$$= \lim_{t\to\infty} \left[-te^{-t} + (-e^{-t}) - (-e^{-0}) \right]$$

$$= \lim_{t\to\infty} \left[-\frac{t}{e^{t}} - \frac{1}{e^t} + 1 \right]$$

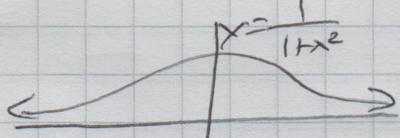
$$= \left(\lim_{t\to\infty} \frac{\overset{\to -\infty}{-t}}{\underset{\to \infty}{e^t}}\right) - \left(\lim_{t\to\infty} \frac{\overset{\to 1}{1}}{\underset{\to\infty}{e^t}}\right) + 1$$

l'Hôpital's Rule

$$= \left(\lim_{t\to\infty} \frac{\frac{d}{dt}(-t)}{\frac{d}{dt}e^t}\right) - 0 + 1$$

$$= \left(\lim_{t\to\infty} \frac{\overset{\to -1}{-1}}{\underset{\to\infty}{e^t}}\right) + 1 = 0 + 1 = 1$$

<u>Terminology</u>: An improper integral that works out a real number is said to <u>converge</u>. If it doesn't, then it is said to <u>diverge</u>.



eg $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ split at some point in between

$$= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \left[\lim_{t\to\infty} \int_{-t}^{0} \frac{1}{1+x^2}\,dx\right] + \left[\lim_{s\to\infty} \int_{0}^{s} \frac{1}{1+x^2}\,dx\right]$$

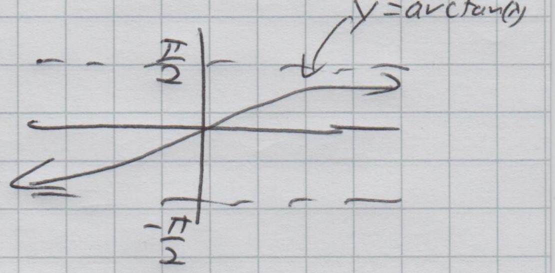
$$= \left[\lim_{t\to\infty} \arctan(x)\Big|_{-t}^{0}\right] + \left[\lim_{s\to\infty} \arctan(x)\Big|_{0}^{s}\right]$$

$$= \left[\lim_{t\to\infty} \left(\arctan(0) - \arctan(-t)\right)\right] + \left[\lim_{s\to\infty} \left(\arctan(s) - \arctan(0)\right)\right]$$

$$= \lim_{t\to\infty}\left[0 - \arctan(-t)\right] + \left[\lim_{s\to\infty}\left(\arctan(s) - 0\right)\right]$$

$$= \left(\lim_{t\to\infty}\left[+\left(+\arctan(t)\right)\right]\right) + \left[\lim_{s\to\infty} \arctan(s)\right]$$

$$= 2\lim_{t\to\infty} \arctan(t)$$



$$= 2\cdot\frac{\pi}{2} = \pi$$

We could have done the following:

$$\int_{-\infty}^{0}\frac{1}{1+x^2}\,dx + \int_{0}^{\infty}\frac{1}{1+x^2}\,dx$$

$$= -\int_{0}^{-\infty}\frac{1}{1+x^2}\,dx + \int_{0}^{\infty}\frac{1}{1+x^2}\,dx$$

$u = -x$

$du = -dx$

x	u
0	0
$-\infty$	∞

$$= \cancel{-}\int_{0}^{\infty}\frac{1}{1+u^2}\cancel{(-1)}\,du + \int_{0}^{\infty}\frac{1}{1+x^2}\,dx = 2\int_{0}^{\infty}\frac{1}{1+x^2}\,dx.$$

Dealing with a vertical asymptote of the integrand:

es $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx = \int_0^1 x^{-1/2}\,dx$ This is undefined at 0 & has a vertical asymptote.

$$= \lim_{t\to 0^+} \int_t^1 x^{-1/2}\,dx$$

$$= \lim_{t\to 0^+} \left.\frac{x^{-1/2+1}}{-\frac{1}{2}+1}\right|_t^1 = \lim_{t\to 0^+} \left.\frac{x^{1/2}}{1/2}\right|_t^1$$

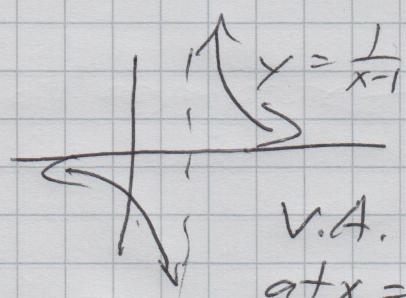
$$= \lim_{t\to 0^+} 2x^{1/2}\Big|_t^1 = \lim_{t\to 0^+} 2\sqrt{x}\Big|_t^1$$

$$= \lim_{t\to 0^+}\left(2\cdot\sqrt{1} - 2\cdot\underset{\substack{\downarrow\\0}}{\sqrt{t}}\right) = 2-0=2$$

If you have v.a. at both ends, split the integral in the middle and handle the limits separately.

Q: What if there is a vertical asymptote in the middle of the integral?

eg $\displaystyle\int_{-1}^{3} \frac{1}{x-1}\,dx$



V.A. at $x=1$

$$= \int_{-1}^{1} \frac{1}{x-1}\,dx + \int_{1}^{3} \frac{1}{x-1}\,dx$$

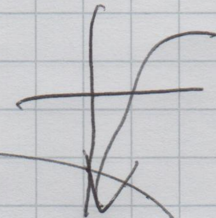
$$= \left(\lim_{t\to 1^-} \int_{-1}^{t} \frac{1}{x-1}\,dx\right) + \left(\lim_{s\to 1^+} \int_{s}^{3} \frac{1}{x-1}\,dx\right)$$

$u = x-1$
$du = dx$

x	u
-1	-2
1	0
3	2

$$= \left(\lim_{t\to 1^-} \int_{-2}^{t-1} \frac{1}{u}\,du\right) + \left(\lim_{s\to 1^+} \int_{s-1}^{2} \frac{1}{u}\,du\right)$$

$$= \lim_{t\to 1^-} \left(\ln|u| \Big|_{-2}^{t-1}\right) + \lim_{s\to 1^+} \left(\ln(u) \Big|_{s-1}^{2}\right)$$



$$= \lim_{t\to 1^-} \left(\ln\underbrace{|t-1|}_{\to 0} - \underbrace{\ln|-2|}_{\ln(2)}\right) + \lim_{s\to 1^+} \left(\underbrace{\ln(2)}_{\ln(2)} - \ln\underbrace{(s-1)}_{\to 0}\right)$$

$$= -\infty - \ln(2) + \ln(2) + (+\infty)$$

$$= \cancel{0}$$

The limits hand you infinities, so the improper integral diverges.