The Comparison Tests

Partly (§11.5)

(Basic) Comparison Test

(in §11.5)

Suppose $0 \le a_n \le b_n$ past some point.

Then (1) if $\sum_{n=0}^{\infty} b_n$ converges,

so does $\sum_{n=0}^{\infty} a_n$,

and (2) if $\sum_{n=0}^{\infty} a_n$ diverges,

so does $\sum_{n=0}^{\infty} b_n$.

Example: $\sum_{n=0}^{\infty} \frac{n^2}{2^n + n^4}$ converge?

We guess it converges and try to simplify the terms in a way that makes them bigger but still keeps convergence

$$\text{if } n \ge 1, \; 0 \le \frac{n^2}{2^n + n^4} \le \frac{n^2}{n^4} = \frac{1}{n^2}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-Test

since $p = 2 - 0 = 2 > 1$.

$\therefore$ $\sum_{n=0}^{\infty} \frac{n^2}{2^n + n^4}$ converges too by the Comparison Test.

Example $\displaystyle\sum_{n=2}^{\infty} \frac{n}{e^n \ln(n)}$ We guess this converges (because the e^n ought to dominate)

When $n \geq 3$, $\ln(n) > 1$, so $\displaystyle\frac{n}{e^n \ln(n)} \leq \frac{n}{e^n} = ne^{-n}$

Apply the Integral Test to $\displaystyle\sum_{n=3}^{\infty} \frac{n}{e^n} = \sum_{n=3}^{\infty} ne^{-n}$.

The corresponding improper integral is

$\displaystyle\int_3^{\infty} xe^{-x}\,dx$ Use integration by parts,

$u = x \qquad v' = e^{-x}$

$u' = 1 \qquad v = -e^{-x}$

$$= \lim_{a\to\infty} \int_3^a xe^{-x}\,dx = \lim_{a\to\infty} \left[-xe^{-x}\Big|_3^a + \int_3^a (+1)e^{-x}\,dx \right]$$

$$= \lim_{a\to\infty} \left[(-ae^{-a}) - (-3e^{-3}) + (-1)e^{-x}\Big|_3^a \right]$$

$$= \lim_{a\to\infty} \left[-ae^{-a} + 3e^{-3} + (-1)e^{-a} - (-1)e^{-3} \right]$$

$a \to \infty$, $e^{-a} \to 0$, $ae^{-a} \to 0$ since exponentials dominate polynomials; $e^{-a} \to 0$

$$= 3e^{-3} + e^{-3} = \frac{4}{e^3}$$

so, by the Integral Test $\displaystyle\sum_{n=3}^{\infty} ne^{-n}$ converges,

so $\displaystyle\sum_{n=2}^{\infty} \frac{n}{e^n \ln(n)}$ converges by the Comparison Test.

Limit Comparison Test (not in the textbook)

Given two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ (which have only have finitely many 0 terms), compute the limit $\lim_{n \to \infty} \left| \frac{a_n}{b_n} \right|$.

(1) If this limit is a real number >0, then the two series both converge or both diverge.

(2) If this limit is 0, then
$\sum_{n=0}^{\infty} a_n$ converges if $\sum_{n=0}^{\infty} b_n$ does.

(3) If this limit is ∞, then
$\sum_{n=0}^{\infty} b_n$ diverges, if $\sum_{n=0}^{\infty} a_n$ does.

Example: $\displaystyle\sum_{n=0}^{\infty} \frac{3^n+4^n}{2^n+5^n}$

Basic Comparison Test:

$$0 \le \frac{3^n+4^n}{2^n+5^n} \le \frac{3^n+4^n}{5^n} \le \frac{2\cdot 4^n}{5^n} = 2\left(\frac{4}{5}\right)^n$$

(since $3^n \le 4^n$)

The series $\displaystyle\sum_{n=0}^{\infty} 2\left(\frac{4}{5}\right)^n$ is a geometric series with first term $a = 2\left(\frac{4}{5}\right)^0 = 2$ & common ratio $|r| = \left|\frac{4}{5}\right| < 1$, so it converges.

Hence the given series converges by the Comparison Test.

Limit Comparison Test:

Look at the dominant terms in the numerator (4^n) and the denominator (5^n) and use those to make $b_n = \frac{4^n}{5^n}$.

$$\lim_{n\to\infty} \left| \frac{\frac{3^n+4^n}{2^n+5^n}}{\frac{4^n}{5^n}} \right| = \lim_{n\to\infty} \frac{3^n+4^n}{2^n+5^n} \cdot \frac{5^n}{4^n}$$

$$= \lim_{n\to\infty} \frac{3^n5^n + 4^n5^n}{2^n4^n + 4^n5^n} \cdot \frac{\frac{1}{4^n5^n}}{\frac{1}{4^n5^n}}$$

$$= \lim_{n\to\infty} \frac{\frac{3^n\cancel{5^n}}{4^n\cancel{5^n}} + \frac{4^n5^n}{4^n5^n}}{\frac{2^n\cancel{4^n}}{\cancel{4^n}5^n} + \frac{4^n5^n}{4^n5^n}} = \lim_{n\to\infty} \frac{\left(\frac{3}{4}\right)^n + 1}{\left(\frac{2}{5}\right)^n + 1} = \frac{0+1}{0+1} = 1$$

Since the limit is a positive real number, we know (by Limit Comparison Test) that $\sum_{n=0}^{\infty} \frac{3^n+4^n}{2^n+5^n}$ and $\sum_{n=0}^{\infty} \frac{4^n}{5^n}$ either both converge or both diverge. But $\sum_{n=0}^{\infty} \frac{4^n}{5^n} = \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n$ is a geometric series with common ratio $r = \frac{4}{5}$. Since $\left|\frac{4}{5}\right| < 1$, this series converges, and hence so does $\sum_{n=0}^{\infty} \frac{3^n+4^n}{2^n+5^n}$.

Added after lecture:

Note: There is a mistake in the statement of the Limit Comparison Test. The absolute value signs in $\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right|$ shouldn't be there and you should only apply the test to series of positive terms. Otherwise conditional convergence messes you up: es $a_n = \frac{(-1)^n}{n}$ & $b_n = \frac{1}{n}$.