

Mathematics 1120H – Calculus II: Integrals and Series

TRENT UNIVERSITY, Summer 2020

Trigonometric Integrals and Substitutions

A Brief Summary

0. A minimal set of trigonometric identities

- $\sin^2(x) + \cos^2(x) = 1$
[Often used in the form $\cos^2(x) = 1 - \sin^2(x)$ or $\sin^2(x) = 1 - \cos^2(x)$.]
- $1 + \tan^2(x) = \sec^2(x)$
[Sometimes used in the form $\sec^2(x) - 1 = \tan^2(x)$.]
- $\sin(2x) = 2 \sin(x) \cos(x)$
- $\cos(2x) = \cos^2(x) - \sin^2(x)$
 $= 2 \cos^2(x) - 1$
 $= 1 - 2 \sin^2(x)$
[Sometimes used in the form $\cos^2(x) = \frac{1}{2} + \frac{1}{2} \cos(2x)$ or $\sin^2(x) = \frac{1}{2} - \frac{1}{2} \cos(2x)$.]

It is also useful to keep in mind that:

- $\sin(x)$ and $\cos(x)$ are *periodic* with period 2π : for any real number x and any integer n , $\sin(x + 2n\pi) = \sin(x)$ and $\cos(x + 2n\pi) = \cos(x)$.
- $\sin(x)$ is an *odd* function, $\sin(-x) = -\sin(x)$ for all x , and $\cos(x)$ is an *even* function, $\cos(-x) = \cos(x)$ for all x .
- Phase shifts are fun: $\sin(x + \frac{\pi}{2}) = \cos(x)$, $\cos(x - \frac{\pi}{2}) = \sin(x)$, $\sin(x \pm \pi) = -\sin(x)$, and $\cos(x \pm \pi) = -\cos(x)$, for all x .

1. Some trigonometric integral reduction formulas

So long as $n \geq 2$, we have:

- $\int \sin^n(x) dx = -\frac{1}{n} \sin^{n-1}(x) \cos(x) + \frac{n-1}{n} \int \sin^{n-2}(x) dx$
- $\int \cos^n(x) dx = \frac{1}{n} \cos^{n-1}(x) \sin(x) + \frac{n-1}{n} \int \cos^{n-2}(x) dx$
- $\int \tan^n(x) dx = \frac{1}{n-1} \tan^{n-1}(x) - \int \tan^{n-2}(x) dx$
- $\int \sec^n(x) dx = \frac{1}{n-1} \tan(x) \sec^{n-2}(x) + \frac{n-2}{n-1} \int \sec^{n-2}(x) dx$
- Just for fun – one usually looks this up as necessary – if we also have $k \geq 2$, then:

$$\begin{aligned} \int \sin^k(x) \cos^n(x) dx &= -\frac{\sin^{k-1}(x) \cos^{n+1}(x)}{k+n} + \frac{k-1}{k+n} \int \sin^{k-2}(x) \cos^n(x) dx \\ &= +\frac{\sin^{k+1}(x) \cos^{n-1}(x)}{k+n} + \frac{n-1}{k+n} \int \sin^k(x) \cos^{n-2}(x) dx \end{aligned}$$

For real obscurity, try to find or compute the corresponding formulas for integrands with mixed $\sec(x)$ and $\tan(x)$, not to mention the various reduction formulas involving $\csc(x)$ and/or $\cot(x)$.

2. Suggestions for trigonometric substitutions

A table of the basic forms:

<i>If you see</i>	<i>try substituting</i>	<i>so</i>	<i>and</i>
$\sqrt{1-x^2}$	$x = \sin(\theta)$	$dx = \cos(\theta) d\theta$	$\cos(\theta) = \sqrt{1-x^2}$
$\sqrt{1+x^2}$	$x = \tan(\theta)$	$dx = \sec^2(\theta) d\theta$	$\sec(\theta) = \sqrt{1+x^2}$
$\sqrt{x^2-1}$	$x = \sec(\theta)$	$dx = \sec(\theta) \tan(\theta) d\theta$	$\tan(\theta) = \sqrt{x^2-1}$

Here is a table of more general forms:

<i>If you see</i>	<i>try substituting</i>	<i>so</i>	<i>and</i>
$\sqrt{a^2-b^2x^2}$	$x = \frac{a}{b} \sin(\theta)$	$dx = \frac{a}{b} \cos(\theta) d\theta$	$\cos(\theta) = \frac{1}{a} \sqrt{a^2-b^2x^2}$
$\sqrt{a^2+b^2x^2}$	$x = \frac{a}{b} \tan(\theta)$	$dx = \frac{a}{b} \sec^2(\theta) d\theta$	$\sec(\theta) = \frac{1}{a} \sqrt{a^2+b^2x^2}$
$\sqrt{b^2x^2-a^2}$	$x = \frac{a}{b} \sec(\theta)$	$dx = \frac{a}{b} \sec(\theta) \tan(\theta) d\theta$	$\tan(\theta) = \frac{1}{a} \sqrt{b^2x^2-a^2}$

3. Handling arbitrary quadratics

How does one handle even more general situations with the square root of an arbitrary quadratic like $\sqrt{px^2+qx+r}$ (where $p \neq 0$) occurs in the integrand? In this case one “completes the square” on the quadratic,

$$\begin{aligned} px^2 + qx + r &= p \left[x^2 + \frac{q}{p}x + \frac{r}{p} \right] = p \left[\left(x + \frac{q}{2p} \right)^2 - \frac{q^2}{4p^2} + \frac{r}{p} \right] \\ &= p \left(x + \frac{q}{2p} \right)^2 + \left(r - \frac{q^2}{4p} \right), \end{aligned}$$

and then uses a substitution like $u = x + \frac{q}{2p}$ to hopefully get a form like one of the “more general” ones above. If you get a form like $\sqrt{-b^2x^2-a^2}$ where what is inside the square root is always negative, you’re out of luck unless you want to start doing calculus with complex numbers.*

4. Be alert to easier alternatives

Do not use the guidelines above without considering possible alternatives: a lot of integrals for which some trigonometric substitution works can also be handled, sometimes more easily, in other ways. For example, $\int x\sqrt{x^2-1} dx$ is probably most easily done with the basic substitution $u = x^2 - 1$.

* Take MATH 3770H in some later year, if you’re interested. Complex analysis has some really fun results, such as Liouville’s Theorem. Where there are plenty of non-constant differentiable functions with bounded output that are defined for all real numbers, such as $\sin(x)$, Liouville’s Theorem asserts that every bounded function that is defined and differentiable for all complex numbers is actually a constant function.