Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2020

Solutions to Assignment #4
A Little Series Algebra

The series $\displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$ is a geometric series with first term $a = 1$ and common ratio $r = x$, and so adds up to $\dfrac{a}{1-r} = \dfrac{1}{1-x}$ when $|r| = |x| < 1$. For questions **1** and **2** you may assume that $|x| < 1$, so that the series adds up nicely.

1. Find a series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ such that $\displaystyle\sum_{n=0}^{\infty} a_n x^n = \left(\sum_{n=0}^{\infty} x^n\right)^2$. *[4]*

SOLUTION. We will take a low-tech brute force approach here and work out $\displaystyle\left(\sum_{n=0}^{\infty} x^n\right)^2$ by multiplying it out and collecting like terms:

$$
\begin{aligned}
\left(\sum_{n=0}^{\infty} x^n\right)^2 &= \left(1 + x + x^2 + x^3 + x^4 + \cdots\right)\left(1 + x + x^2 + x^3 + x^4 + \cdots\right) \\
&= 1\left(1 + x + x^2 + x^3 + x^4 + \cdots\right) \\
&\qquad + x\left(1 + x + x^2 + x^3 + x^4 + \cdots\right) \\
&\qquad\qquad + x^2\left(1 + x + x^2 + x^3 + x^4 + \cdots\right) \\
&\qquad\qquad\qquad + x^3\left(1 + x + x^2 + x^3 + x^4 + \cdots\right) \\
&\qquad\qquad\qquad\qquad \vdots \\
&= 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots \\
&\qquad + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots \\
&\qquad\qquad + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots \\
&\qquad\qquad\qquad + x^3 + x^4 + x^5 + x^6 + \cdots \\
&\qquad\qquad\qquad\qquad \vdots \\
&= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + \cdots
\end{aligned}
$$

The desired series is therefore $\displaystyle\sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$. $\square$

2. Find a series $\displaystyle\sum_{n=0}^{\infty} b_n x^n$ such that $\displaystyle\left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = 1$. *[1]*

SOLUTION. Recall that the geometric series sum formula tells us that $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$. This gives us a clue, namely that

$$\left(1 + x + x^2 + x^3 + x^4 + \cdots\right)(1 - x) = 1\,.$$

Since $1 - x = 1 - x + 0x^2 + 0x^3 + 0x^4 + \cdots$, the series $\sum_{n=0}^{\infty} b_n x^n$ with $b_0 = 1$, $b_1 = -1$, and $b_n - 0$ for $n \geq 2$ does the job. Note that pretty much everyone who isn't a total mathochist would write this series simply as $1 - x$. □

Recall from Assignment #3 that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$. This series actually converges for all x, as we shall see later.

3. Find a series $\sum_{n=0}^{\infty} c_n x^n$ such that $\sum_{n=0}^{\infty} c_n x^n = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)^2$. *[3]*

SOLUTION. We exploit the fact that we know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x repeatedly:

$$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)^2 = (e^x)^2 = e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$$

Thus $\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$, *i.e.* $c_n = \frac{2^n}{n!}$ for all $n \geq 0$, is the series we're looking for. □

4. Find a series $\sum_{n=0}^{\infty} d_n x^n$ such that $\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} d_n x^n\right) = 1$. *[2]*

SOLUTION. We exploit the fact that we know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x again:

$$1 = e^0 = e^{x-x} = e^x e^{-x} = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}\right)$$

Thus $\sum_{n=0}^{\infty} d_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$, *i.e.* $d_n = \frac{(-1)^n}{n!}$ for all $n \geq 0$, is the series we're looking for. ■