Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2020

Solutions to Assignment #2
Integration Challenge

Two of the three integrals below require substitution to work out, and two of the three integrals below require integration by parts to work out. Show all the major steps in your solutions.

1. Compute $\displaystyle\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx$. *[3]*

SOLUTION. We will use the substitution $u = e^x + 1$, so $du = e^x\,dx$. Note that $e^x = u - 1$ and that $e^{2x} = (e^x)^2$.

$$\begin{aligned}\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx &= \int \frac{(e^x)^2}{\sqrt{e^x+1}}\,dx = \int \frac{e^x}{\sqrt{e^x+1}}\,e^x\,dx = \int \frac{u-1}{\sqrt{u}}\,du \\ &= \int \left(\sqrt{u} - \frac{1}{\sqrt{u}}\right)du = \int \left(u^{1/2} - u^{-1/2}\right)du = \frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2} + C \\ &= \frac{2}{3}u^{3/2} - 2u^{1/2} + C = \frac{2}{3}\left(e^x+1\right)^{3/2} - 2\left(e^x+1\right)^{1/2} + C \qquad \square\end{aligned}$$

2. Compute $\displaystyle\int_1^{e^{\pi/2}} \cos\left(\ln(x)\right)\,dx$. *[4]*

SOLUTION. So as to avoid having to muck about with the limits of integration as much as possible, we will work out the anti-derivative of $\cos\left(\ln(x)\right)$ and then use it to evaluate the given definite integral.

(With substitution.) The difficulty in $\displaystyle\int \cos\left(\ln(x)\right)\,dx$ is that we are stuffing $\ln(x)$ into $\cos(\ )$, so will try to simplify the integrand by using the substitution $w = \ln(x)$. As we lack the derivative of $\ln(x)$, namely $\frac{1}{x}$, in the integrand, it is better to think of the substitution $w = \ln(x)$ in reverse, namely as $x = e^w$, so $dx = e^w\,dw$. We then have:

$$\int \cos\left(\ln(x)\right)\,dx = \int \cos(w)\cdot e^w\,dw = \int e^w\cos(w)\,dw$$

The integral can be worked out using integration by parts, and was in the lecture on integration by parts, so we'll skip the actual calculation and head straight to its conclusion:

$$\begin{aligned}\int \cos\left(\ln(x)\right)\,dx = \int e^w\cos(w)\,dw &= \frac{e^w}{2}\left[\cos(w)+\sin(w)\right] + C \\ &= \frac{e^{\ln(x)}}{2}\left[\cos\left(\ln(x)\right)+\sin\left(\ln(x)\right)\right] + C \\ &= \frac{x}{2}\left[\cos\left(\ln(x)\right)+\sin\left(\ln(x)\right)\right] + C\end{aligned}$$

(Without substitution.) We will use integration by parts only. As the integrand is not naturally a product of two functions, we will use the dummy product trick.

$$\begin{aligned}
\int \cos(\ln(x))\,dx &= \int 1\cdot\cos(\ln(x))\,dx && \begin{array}{l} u=\cos(\ln(x)) \text{ and } v'=1\text{, so} \\ u'=-\sin(\ln(x))\cdot\dfrac{1}{x} \text{ and } v=x \end{array} \\
&= \cos(\ln(x))\cdot x - \int(-1)\sin(\ln(x))\cdot\frac{1}{x}\cdot x\,dx \\
&= x\cos(\ln(x)) + \int \sin(\ln(x))\,dx && \begin{array}{l} s=\sin(\ln(x)) \text{ and } t'=1\text{, so} \\ s'=\cos(\ln(x))\cdot\dfrac{1}{x} \text{ and } v=x \end{array} \\
&= x\cos(\ln(x)) + x\sin(\ln(x)) - \int\cos(\ln(x))\cdot\frac{1}{x}\cdot x\,dx \\
&= x\cos(\ln(x)) + x\sin(\ln(x)) - \int\cos(\ln(x))\,dx
\end{aligned}$$

Comparing the beginning and the end of the calculation, we can solve for the antiderivative:

$$\begin{aligned}
2\int\cos(\ln(x))\,dx &= x\cos(\ln(x)) + x\sin(\ln(x)) \\
\Longrightarrow \int\cos(\ln(x))\,dx &= \frac{x}{2}\left[\cos(\ln(x)) + \sin(\ln(x))\right] + C
\end{aligned}$$

Either way, we now have:

$$\begin{aligned}
\int_1^{e^{\pi/2}}\cos(\ln(x))\,dx &= \frac{x}{2}\left[\cos(\ln(x)) + \sin(\ln(x))\right]\Big|_1^{e^{\pi/2}} \\
&= \frac{e^{\pi/2}}{2}\left[\cos\left(\ln(e^{\pi/2})\right) + \sin\left(\ln(e^{\pi/2})\right)\right] - \frac{1}{2}\left[\cos(\ln(1)) + \sin(\ln(1))\right] \\
&= \frac{1}{2}e^{\pi/2}\left[\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)\right] - \frac{1}{2}\left[\cos(0)+\sin(0)\right] \\
&= \frac{1}{2}e^{\pi/2}[0+1] - \frac{1}{2}[1+0] = \frac{1}{2}\left(e^{\pi/2}-1\right) \qquad \square
\end{aligned}$$

3. Compute $\displaystyle\int\frac{1}{(x^2+1)^3}\,dx$. *[3]*

SOLUTION. Just to play with techniques of integration, we'll do this in two different ways.

(Derive and use a reduction formula.) We will first derive the reduction formula

$$\int\frac{1}{(x^2+1)^n}\,dx = \frac{1}{2n-2}\cdot\frac{x}{(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}\int\frac{1}{(x^2+1)^{n-1}}\,dx$$

and then apply it to the given indefinite integral, partly to show that reduction formulas need not just be for trigonometric functions. Note that for the formula to work, we will need $n > 1$. (Why?)

To get the reduction formula, observe first that:

$$\begin{aligned}\int \frac{1}{(x^2+1)^n}\,dx &= \int \frac{x^2+1-x^2}{(x^2+1)^n}\,dx = \int \frac{x^2+1}{(x^2+1)^n}\,dx - \int \frac{x^2}{(x^2+1)^n}\,dx \\ &= \int \frac{1}{(x^2+1)^{n-1}}\,dx - \int \frac{x^2}{(x^2+1)^n}\,dx\end{aligned}$$

We will tackle the last integral using integration by parts, with $u = x$ and $v' = \dfrac{x}{(x^2+1)^n}$, so $u' = 1$ and

$$\begin{aligned}v &= \int \frac{x}{(x^2+1)^n}\,dx \qquad \text{Substitute } w = x^2+1\text{, so } dw = 2x\,dx \text{ and } \frac{1}{2}\,dw = x\,dx. \\ &= \int \frac{1}{w^n}\cdot\frac{1}{2}\,dw = \frac{1}{2}\int w^{-n}\,dw = \frac{1}{2}\cdot\frac{w^{-n+1}}{-n+1} = -\frac{1}{2n-2}\cdot\frac{1}{w^{n-1}} \\ &= -\frac{1}{2n-2}\cdot\frac{1}{(x^2+1)^{n-1}}\,.\end{aligned}$$

(We ignore the usual $+C$ in an antiderivative because we're putting this to use integrating by parts.) It now follows that

$$\begin{aligned}\int \frac{1}{(x^2+1)^n}\,dx &= \int \frac{1}{(x^2+1)^{n-1}}\,dx - \int \frac{x^2}{(x^2+1)^n}\,dx \\ &= \int \frac{1}{(x^2+1)^{n-1}}\,dx - \left[x\cdot(-1)\frac{1}{2n-2}\cdot\frac{1}{(x^2+1)^{n-1}}\right. \\ &\qquad\qquad \left. - \int 1\cdot(-1)\frac{1}{2n-2}\cdot\frac{1}{(x^2+1)^{n-1}}\,dx\right] \\ &= \int \frac{1}{(x^2+1)^{n-1}}\,dx + \frac{1}{2n-2}\cdot\frac{x}{(x^2+1)^{n-1}} - \frac{1}{2n-2}\int \frac{1}{(x^2+1)^{n-1}}\,dx \\ &= \frac{1}{2n-2}\cdot\frac{x}{(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}\int \frac{1}{(x^2+1)^{n-1}}\,dx\,,\end{aligned}$$

which is the reduction formula we wanted. (Whew! :-)

It remains to apply the formula to the given integral:

$$\begin{aligned}\int \frac{1}{(x^2+1)^3}\,dx &= \frac{1}{2\cdot 3-2}\cdot\frac{x}{(x^2+1)^{3-1}} + \frac{2\cdot 3-3}{2\cdot 3-2}\int \frac{1}{(x^2+1)^{3-1}}\,dx \\ &= \frac{1}{4}\cdot\frac{x}{(x^2+1)^2} + \frac{3}{4}\int \frac{1}{(x^2+1)^2}\,dx\end{aligned}$$

… and use it again one more time:

$$\begin{aligned}
\int \frac{1}{\left(x^2+1\right)^3}\,dx &= \frac{1}{4}\cdot\frac{x}{\left(x^2+1\right)^2}+\frac{3}{4}\int\frac{1}{\left(x^2+1\right)^2}\,dx \\
&= \frac{1}{4}\cdot\frac{x}{\left(x^2+1\right)^2}+\frac{3}{4}\left[\frac{1}{2\cdot 2-2}\cdot\frac{x}{\left(x^2+1\right)^{2-1}}\right. \\
&\qquad\qquad\qquad\qquad\left.+\frac{2\cdot 2-3}{2\cdot 2-2}\int\frac{1}{\left(x^2+1\right)^{2-1}}\,dx\right] \\
&= \frac{1}{4}\cdot\frac{x}{\left(x^2+1\right)^2}+\frac{3}{4}\left[\frac{1}{2}\cdot\frac{x}{x^2+1}+\frac{1}{2}\int\frac{1}{x^2+1}\,dx\right] \\
&= \frac{1}{4}\cdot\frac{x}{\left(x^2+1\right)^2}+\frac{3}{8}\cdot\frac{x}{x^2+1}+\frac{3}{8}\int\frac{1}{x^2+1}\,dx \\
&= \frac{1}{4}\cdot\frac{x}{\left(x^2+1\right)^2}+\frac{3}{8}\cdot\frac{x}{x^2+1}+\frac{3}{8}\arctan(x)+C
\end{aligned}$$

Proving the reduction formula first is probably overkill, but it's a formula that can come in handy when integrating partial fractions, as we shall soon see in "class".

(Trigonometric substitution.) This time we will use the trigonometric substitution $x=\tan(t)$, so $dx=\sec^2(t)\,dt$. We will eventually use the reduction formula for integrals of powers of $\cos(t)$ (twice).

$$\begin{aligned}
\int \frac{1}{\left(x^2+1\right)^3}\,dx &= \int\frac{1}{\left(\tan^2(t)+1\right)^3}\sec^2(t)\,dt = \int\frac{\sec^2(t)}{\left(\sec^2(t)\right)^3}\,dt = \int\frac{\sec^2(t)}{\sec^6(t)}\,dt \\
&= \int\frac{1}{\sec^4(t)}\,dt = \int\left(\frac{1}{\sec(t)}\right)^4\,dt = \int\cos^4(t)\,dt \\
&= \frac{1}{4}\cos^3(t)\sin(t)+\frac{3}{4}\int\cos^2(t)\,dt \\
&= \frac{1}{4}\cos^3(t)\sin(t)+\frac{3}{4}\left[\frac{1}{2}\cos(t)\sin(t)+\frac{1}{2}\int\cos^0(t)\,dt\right] \\
&= \frac{1}{4}\cos^3(t)\sin(t)+\frac{3}{8}\cos(t)\sin(t)+\frac{3}{8}\int 1\,dt \\
&= \frac{1}{4}\cos^3(t)\sin(t)+\frac{3}{8}\cos(t)\sin(t)+\frac{3}{8}t+C
\end{aligned}$$

It remains to put the antiderivative in terms of x. We have $x=\tan(t)$, so $t=\arctan(x)$, and

$$\cos(t)=\frac{1}{\sec(t)}=\frac{1}{\sqrt{\sec^2(t)}}=\frac{1}{\sqrt{\tan^2(t)+1}}=\frac{1}{\sqrt{x^2+1}}$$

and

$$
\begin{aligned}
\sin(t) = \sqrt{1-\cos^2(t)} &= \sqrt{1-\left(\frac{1}{\sqrt{x^2+1}}\right)^2} \\
&= \sqrt{\frac{x^2+1}{x^2+1}-\frac{1}{x^2+1}} = \sqrt{\frac{x^2}{x^2+1}} = \frac{x}{\sqrt{x^2+1}}\,.
\end{aligned}
$$

Thus:

$$
\begin{aligned}
\int \frac{1}{(x^2+1)^3}\,dx &= \frac{1}{4}\cos^3(t)\sin(t)+\frac{3}{8}\cos(t)\sin(t)+\frac{3}{8}t+C \\
&= \frac{1}{4}\left(\frac{1}{\sqrt{x^2+1}}\right)^3\frac{x}{\sqrt{x^2+1}}+\frac{3}{8}\cdot\frac{1}{\sqrt{x^2+1}}\cdot\frac{x}{\sqrt{x^2+1}}+\frac{3}{8}\arctan(x)+C \\
&= \frac{1}{4}\cdot\frac{x}{(x^2+1)^2}+\frac{3}{8}\cdot\frac{x}{x^2+1}+\frac{3}{8}\arctan(x)+C \qquad \blacksquare
\end{aligned}
$$