Mathematics 1120H – Calculus II: Integrals and Series

Trent University, Summer 2020

Solutions to Quiz #6

We know from lecture that the Taylor series at 0 (otherwise known as the MacLaurin series) of $\cos(x)$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} .$$

1. As was done in the lecture for $\cos(x)$, use Taylor's formula to find the Taylor series at 0 of $\sin(x)$ and determine its interval of convergence. *[2.5]*

Solution. We grind out the derivatives at 0 of $f(x) = \sin(x)$ and look for a pattern to plug into Taylor's formula:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin(x)$	0
1	$\cos(x)$	1
2	$-\sin(x)$	0
3	$-\cos(x)$	-1
4	$\sin(x)$	0
5	$\cos(x)$	1
6	$-\sin(x)$	0
7	$-\cos(x)$	-1
8	$\sin(x)$	0
$\vdots$	$\vdots$	$\vdots$

At all even n, we have $f^{(n)}(0) = 0$; at odd values of n, say $n = 2k+1$ where $k \geq 0$, we have $f^{(n)}(0) = 1$ if $k = 0, 2, 4, \dots$ and $f^{(n)}(0) = -1$ if $k = 1, 3, 5, \dots$, *i.e.* $f^{(2k+1)}(0) = (-1)^k$. It follows that the Taylor series at 0 of $f(x) = \sin(x)$ is:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n &= \sum_{k=0}^{\infty} \frac{f^{(2k+1)}(0)}{(2k+1)!} x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \\ &= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\end{aligned}$$

It remains to determine the the interval of convergence of this series. As usual we appeal to the Ratio Test first:

$$\begin{aligned}\lim_{k\to\infty} \left| \frac{\frac{(-1)^{k+1}}{(2(k+1)+1)!} x^{2(k+1)+1}}{\frac{(-1)^k}{(2k+1)!} x^{2k+1}} \right| &= \lim_{k\to\infty} \left| \frac{\frac{(-1)^{k+1}}{(2k+3)!} x^{2k+3}}{\frac{(-1)^k}{(2k+1)!} x^{2k+1}} \right| = \lim_{k\to\infty} \left| \frac{(-1)^{k+1} x^{2k+3}}{(2k+3)!} \cdot \frac{(2k+1)!}{(-1)^k x^{2k+1}} \right| \\ &= \lim_{k\to\infty} \left| \frac{(-1)x^2}{(2k+3)(2k+2)} \right| = x^2 \lim_{k\to\infty} \frac{1}{(2k+3)(2k+2)} \\ &= x^2 \cdot 0 = 0\end{aligned}$$

Since, no matter what value x has, we get a limit of in the Ratio Test and $0 < 1$, the series converges for all x, *i.e.* the interval of convergence of this series is $(-\infty, \infty)$. □

2. Find the Taylor series at 0 of $\sin(x)$ without (directly) using Taylor's formula. *[1]*

SOLUTION. Since antiderivative of $\cos(x)$ is $\sin(x)$, it follows that the antiderivative of the Taylor series at 0 for $\cos(x)$ is (up to a constant) the Taylor series at 0 for $\sin(x)$:

$$\int \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \right) dx = \sum_{n=0}^{\infty} \int \left(\frac{(-1)^n}{(2n)!} x^{2n} \right) dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{x^{2n+1}}{2n+1}$$
$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

The constant of integration, C, can be solved for because the function $\sin(x)$ and its Taylor series at 0 must equal each other at $x = 0$:

$$0 = \sin(0) = C + \sum_{n=0}^{\infty} \frac{(-1)^n 0^{2n+1}}{(2n+1)!} = C + 0 = C$$

Thus the Taylor series at 0 of $\sin(x)$ is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$. □

3. Find the Taylor series at 0 of $f(x) = \sin(x) + x\cos(x)$. *[1.5]*

SOLUTION. Recall that if we have a power series at a equal to a function, that power series is the Taylor series at a of the function. Since we know the Taylor series at 0 of $\sin(x)$ and $\cos(x)$, and these series are equal to the functions they came from when they converge (like most Taylor series), the Taylor series at 0 of $f(x) = \sin(x) + x\cos(x)$ is given by:

$$\begin{aligned}
f(x) &= \sin(x) + x\cos(x) \\
&= \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right) + x \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \right) \\
&= \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right) + \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!} \right) \\
&= \sum_{n=0}^{\infty} \left(\frac{(-1)^n x^{2n+1}}{(2n+1)!} + \frac{(-1)^n x^{2n+1}}{(2n)!} \right) \\
&= \sum_{n=0}^{\infty} \left(\frac{(-1)^n x^{2n+1}}{(2n+1)!} + \frac{(-1)^n x^{2n+1}(2n+1)}{(2n)!(2n+1)} \right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+2) x^{2n+1}}{(2n+1)!} \qquad \blacksquare
\end{aligned}$$