Mathematics 1120H – Calculus II: Integrals and Series

Trent University, Summer 2020 (S62)

Solutions to the Take-Home Final Examination

Instructions

- You may consult your notes, handouts, and textbook from this course and any other math courses you have taken or are taking now. You may also use a calculator. However, you may not consult any other source, or give or receive any other aid, except for asking the instructor to clarify instructions or questions.
- Please submit an electronic copy of your solutions, preferably as a single pdf (a scan of handwritten solutions should be fine), via the Assignment module on Blackboard. If that doesn't work, please email your solutions to the intructor. *Show all your work!*
- Do all three (3) of Parts **I** – **III**, and, if you wish, Part **IV** as well.

Part I. Do both of **1** and **2**. *[40 = 2×20 each]*

1. Compute the integrals in any four (4) of **a** – **f**. *[20 = 4×5 each]*

a. $\displaystyle\int_0^{\pi/2} \sin(x)\sqrt{1+\cos^2(x)}\,dx$ **b.** $\displaystyle\int \frac{\ln(\ln(x))}{x}\,dx$ **c.** $\displaystyle\int \frac{x}{\sqrt{4-x^2}}\,dx$

d. $\displaystyle\int_{-1}^{1} \frac{1+\arctan^2(x)}{1+x^2}\,dx$ **e.** $\displaystyle\int_0^1 x\arctan(x)\,dx$ **f.** $\displaystyle\int \frac{1}{\sqrt{4+x^2}}\,dx$

Solutions. **a.** We will use the substitution $u=\cos(x)$, so $du=(-1)\sin(x)\,dx$ and $\sin(x)\,dx=(-1)\,du$, and change the limits as we go along: $\begin{matrix} x & 0 & \pi/2 \\ u & 1 & 0 \end{matrix}$.

$$\int_0^{\pi/2} \sin(x)\sqrt{1+\cos^2(x)}\,dx = \int_1^0 \sqrt{1+u^2}\,(-1)\,du = \int_0^1 \sqrt{1+u^2}\,du$$

Substitute again, this time with $u=\tan(t)$, so $du=\sec^2(t)\,dt$, and change the limits accordingly: $\begin{matrix} u & 0 & 1 \\ t & 0 & \pi/4 \end{matrix}$.

$$\begin{aligned}
\int_0^1 \sqrt{1+u^2}\,du &= \int_0^{\pi/4} \sqrt{1+\tan^2(t)}\,\sec^2(t)\,dt = \int_0^{\pi/4} \sqrt{\sec^2(t)}\,\sec^2(t)\,dt \\
&= \int_0^{\pi/4} \sec(t)\sec^2(t)\,dt = \int_0^{\pi/4} \sec^3(t)\,dt \quad \text{Reduction formula:} \\
&= \frac{1}{3-1}\tan(t)\sec^{3-2}(t)\Big|_0^{\pi/4} + \frac{3-2}{3-1}\int_0^{\pi/4} \sec^{3-2}(t)\,dt \\
&= \frac{1}{2}\tan(t)\sec(t)\Big|_0^{\pi/4} + \frac{1}{2}\ln(\tan(t)+\sec(t))\Big|_0^{\pi/4} \\
&= \frac{1}{2}\cdot 1\cdot\sqrt{2} - \frac{1}{2}\cdot 0\cdot 1 + \frac{1}{2}\ln\left(1+\sqrt{2}\right) - \frac{1}{2}\ln(0+1) \\
&= \frac{\sqrt{2}}{2} - 0 + \frac{1}{2}\ln\left(1+\sqrt{2}\right) - 0 = \frac{1}{\sqrt{2}} + \frac{1}{2}\ln\left(1+\sqrt{2}\right) \qquad \square
\end{aligned}$$

b. First, substitute $u = \ln(x)$, so $du = \frac{1}{x}\, dx$.

$$\int \frac{\ln(\ln(x))}{x}\, dx = \int \ln(u)\, du \quad \text{Now use integration by parts with } s = \ln(u) \text{ and } t' = 1\text{, so } s' = \tfrac{1}{u} \text{ and } t = u.$$

$$= \ln(u) \cdot u - \int \frac{1}{u} \cdot u\, du = u\ln(u) - \int 1\, du = u\ln(u) - u + C$$

$$= \ln(x)\ln(\ln(x)) - \ln(x) + C \qquad \square$$

c. *(Substitution)* Substitute $w = 4 - x^2$, so $dw = -2x\, dx$ and $x\, dx = \left(-\frac{1}{2}\right) dw$.

$$\int \frac{x}{\sqrt{4-x^2}}\, dx = \int \frac{1}{\sqrt{w}} \left(-\frac{1}{2}\right) dw = -\frac{1}{2} \int w^{-1/2}\, dw = -\frac{1}{2} \cdot \frac{w^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= -\frac{1}{2} \cdot \frac{w^{1/2}}{1/2} + C = -w^{1/2} + C = -\left(4 - x^2\right)^{1/2} + C$$

$$= -\sqrt{4 - x^2} + C \qquad \square$$

c. *(Trigonometric substitution)* Substitute $x = 2\sin(\theta)$, so $dx = 2\cos(\theta)\, d\theta$. Note that $\sin(\theta) = \frac{x}{2}$ and $\cos(\theta) = \sqrt{1 - \sin^2(\theta)} = \sqrt{1 - \frac{x^2}{4}}$.

$$\int \frac{x}{\sqrt{4-x^2}}\, dx = \int \frac{2\sin(\theta)}{\sqrt{4 - 4\sin^2(\theta)}} 2\cos(\theta)\, d\theta = \int \frac{4\sin(\theta)\cos(\theta)}{2\sqrt{1 - \sin^2(\theta)}}\, d\theta$$

$$= \int \frac{2\sin(\theta)\cos(\theta)}{\cos(\theta)}\, d\theta = \int 2\sin(\theta)\, d\theta = -2\cos(\theta) + C$$

$$= -2\sqrt{1 - \frac{x^2}{4}} + C = -\sqrt{4 - x^2} + C \qquad \square$$

d. Substitute $u = \arctan(x)$, so $du = \dfrac{1}{1+x^2}$ and change the limits: $\begin{array}{ccc} x & -1 & 1 \\ u & -\pi/4 & \pi/4 \end{array}$.

$$\int_{-1}^{1} \frac{1 + \arctan^2(x)}{1 + x^2}\, dx = \int_{-\pi/4}^{\pi/4} \left(1 + u^2\right)\, du = \left(u + \frac{u^3}{3}\right)\Bigg|_{-\pi/4}^{\pi/4}$$

$$= \left(\frac{\pi}{4} + \frac{1}{3}\left(\frac{\pi}{4}\right)^3\right) - \left(-\frac{\pi}{4} + \frac{1}{3}\left(-\frac{\pi}{4}\right)^3\right)$$

$$= \left(\frac{\pi}{4} + \frac{\pi^3}{192}\right) - \left(-\frac{\pi}{4} - \frac{\pi^3}{192}\right) = \frac{\pi}{2} + \frac{\pi^3}{96} \qquad \square$$

e. Integrate by parts with $u = \arctan(x)$ and $v' = x$, so $u' = \dfrac{1}{1+x^2}$ and $v = \dfrac{x^2}{2}$.

$$\begin{aligned}
\int_0^1 x\arctan(x)\,dx &= \frac{x^2}{2}\arctan(x)\Big|_0^1 - \int_0^1 \frac{x^2}{2}\cdot\frac{1}{1+x^2}\,dx \\
&= \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\cdot 0 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}\,dx \\
&= \frac{\pi}{8} - \frac{1}{2}\int_0^1 \frac{1+x^2-1}{1+x^2}\,dx \\
&= \frac{\pi}{8} - \frac{1}{2}\int_0^1 \frac{1+x^2}{1+x^2}\,dx - \frac{1}{2}\int_0^1 \frac{-1}{1+x^2}\,dx \\
&= \frac{\pi}{8} - \frac{1}{2}\int_0^1 1\,dx + \frac{1}{2}\arctan(x)\Big|_0^1 \\
&= \frac{\pi}{8} - \frac{1}{2}x\Big|_0^1 + \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\cdot 0 \\
&= \frac{\pi}{8} - \left(\frac{1}{2}\cdot 1 - \frac{1}{2}\cdot 0\right) + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2} \qquad \square
\end{aligned}$$

f. Substitute $x = 2\tan(\theta)$, so $dx = 2\sec^2(\theta)$. Note that $\tan(\theta) = \frac{x}{2}$ and $\sec(\theta) = \sqrt{1+\tan^2(\theta)} = \sqrt{1+\frac{x^2}{4}}$.

$$\begin{aligned}
\int \frac{1}{\sqrt{4+x^2}}\,dx &= \int \frac{1}{\sqrt{4+4\tan^2(\theta)}}\,2\sec^2(\theta)\,d\theta = \int \frac{2\sec^2(\theta)}{2\sqrt{1+\tan^2(\theta)}}\,d\theta \\
&= \int \frac{\sec^2(\theta)}{\sqrt{\sec^2(\theta)}}\,d\theta = \int \frac{\sec^2(\theta)}{\sec(\theta)}\,d\theta = \int \sec(\theta)\,d\theta \\
&= \ln\left(\tan(\theta) + \sec(\theta)\right) + C = \ln\left(\frac{x}{2} + \sqrt{1+\frac{x^2}{4}}\right) + C \qquad \blacksquare
\end{aligned}$$

2. Determine whether the series converges in any four (4) of **a** – **f**. *[20 = 4 × 5 each]*

$$\textbf{a.}\ \sum_{n=0}^{\infty} \frac{2^n - 3^n}{4^n + (-1)^n} \qquad \textbf{b.}\ \sum_{n=0}^{\infty} (-3)^{-n} e^n \qquad \textbf{c.}\ \sum_{n=1}^{\infty} \frac{\ln(n)}{n}$$

$$\textbf{d.}\ \sum_{n=0}^{\infty} \frac{\sin(n) + \cos(n)}{n^3 + n^2 + n + 1} \qquad \textbf{e.}\ \sum_{n=1}^{\infty} \frac{n^n}{n!} \qquad \textbf{f.}\ \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

SOLUTIONS. **a.** *(Limit Comparison Test)* We will show that the given series is absolutely convergent using the Limit Comparison Test. Since the dominant term in the numerator of the given series as n increases is 3^n and the dominant term in the denominator is 4^n, we will compare $\displaystyle\sum_{n=0}^{\infty} \left| \frac{2^n - 3^n}{4^n + (-1)^n} \right|$ to the geometric series $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{4^n} = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\left|\frac{2^n - 3^n}{4^n + (-1)^n}\right|}{\frac{3^n}{4^n}} &= \lim_{n\to\infty} \left| \frac{2^n - 3^n}{4^n + (-1)^n} \right| \cdot \frac{\frac{1}{3^n}}{\frac{1}{4^n}} = \lim_{n\to\infty} \left| \frac{\frac{2^n}{3^n} - \frac{3^n}{3^n}}{\frac{4^n}{4^n} + \frac{(-1)^n}{4^n}} \right| \\
&= \lim_{n\to\infty} \left| \frac{\left(\frac{2}{3}\right)^n - 1}{1 + \left(-\frac{1}{4}\right)^n} \right| = \left| \frac{0 - 1}{1 - 0} \right| = |-1| = 1\,,
\end{aligned}$$

since $\left(\frac{2}{3}\right)^n \to 0$ and $\left(-\frac{1}{4}\right)^n \to 0$ as $n \to \infty$. As the comparison limit of 1 is a positive real number, the series $\displaystyle\sum_{n=0}^{\infty} \left| \frac{2^n - 3^n}{4^n + (-1)^n} \right|$ and $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{4^n} = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$ both converge or both diverge by the Limit Comparison Test. Since the geometric series $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$ converges, having a common ratio with $|r| = \frac{3}{4} < 1$, it follows that $\displaystyle\sum_{n=0}^{\infty} \left| \frac{2^n - 3^n}{4^n + (-1)^n} \right|$ converges too.

Thus the given series, $\displaystyle\sum_{n=0}^{\infty} \frac{2^n - 3^n}{4^n + (-1)^n}$, converges absolutely, and hence converges. □

a. *(Basic Comparison Test)* We will show that the given series is absolutely convergent using the Basic Comparison Test. Note that since $2^n < 3^n$ and $4^n + (-1)^n \geq 4^n - 1 > 4^{n-1}$ for all $n \geq 1$, we have

$$0 \leq \left| \frac{2^n - 3^n}{4^n + (-1)^n} \right| \leq \frac{2^n + 3^n}{4^n - 1} < \frac{2 \cdot 3^n}{4^{n-1}} = \frac{2 \cdot 3 \cdot 3^{n-1}}{4^{n-1}} = 6 \left(\frac{3}{4}\right)^{n-1}$$

for all $n \geq 1$. Since $\displaystyle\sum_{n=0}^{\infty} 6 \left(\frac{3}{4}\right)^{n-1}$ is a geometric series whose common ratio satisfies $|r| = \frac{3}{4} < 1$, it converges. By the Basic Comparison Test it follows that $\displaystyle\sum_{n=0}^{\infty} \left| \frac{2^n - 3^n}{4^n + (-1)^n} \right|$.

Thus the given series, $\displaystyle\sum_{n=0}^{\infty} \frac{2^n - 3^n}{4^n + (-1)^n}$, converges absolutely, and hence converges. □

b. $\displaystyle\sum_{n=0}^{\infty}(-3)^{-n}e^n = \sum_{n=0}^{\infty}\frac{e^n}{(-3)^n} = \sum_{n=0}^{\infty}\left(-\frac{e}{3}\right)^n$ is a geometric series with common ratio $r=-\dfrac{e}{3}$. As $e = 2.718\cdots < 3$, we have $|r| = \left|-\dfrac{e}{3}\right| = \dfrac{e}{3} < 1$, so the series converges. □

NOTE: The series in **b** can also be shown to converge very easily using the Ratio Test or the Root Test, and pretty easily using the Alternating Series Test.

c. $f(x) = \dfrac{\ln(x)}{x}$ is non-negative and continuous (and hence integrable) for all $x \geq 1$. By the Integral Test, it follows that the series $\displaystyle\sum_{n=1}^{\infty}\frac{\ln(n)}{n}$ and the improper integral $\displaystyle\int_1^{\infty}\frac{\ln(x)}{x}\,dx$ both converge or both diverge. We compute the integral:

$$\int_1^{\infty}\frac{\ln(x)}{x}\,dx = \lim_{a\to\infty}\int_1^a\frac{\ln(x)}{x}\,dx \qquad \begin{array}{l}\text{Substitute } u=\ln(x)\text{, so } du=\frac{1}{x}\,dx,\\ \text{and change the limits accordingly:}\end{array} \quad \begin{array}{ccc} x & 1 & a \\ u & 0 & \ln(a)\end{array}$$

$$= \lim_{a\to\infty}\int_0^{\ln(a)}u\,du = \lim_{a\to\infty}\left.\frac{u^2}{2}\right|_0^{\ln(a)} = \lim_{a\to\infty}\left(\frac{(\ln(a))^2}{2}-\frac{0^2}{2}\right) = \infty\,,$$

since $\ln(a)\to\infty$ as $a\to\infty$. Since the improper integral does not work out to a real number, *i.e.* it diverges, the given series also diverges. □

d. We will use the Basic Comparison Test to show the series $\displaystyle\sum_{n=0}^{\infty}\frac{\sin(n)+\cos(n)}{n^3+n^2+n+1}$ converges absolutely. Note that

$$0 \leq \left|\frac{\sin(n)+\cos(n)}{n^3+n^2+n+1}\right| \leq \frac{|\sin(n)|+|\cos(n)|}{n^3+n^2+n+1} \leq \frac{1+1}{n^3} = \frac{2}{n^3}$$

for all $n \geq 1$. Since $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n^3} = 2\sum_{n=1}^{\infty}\frac{1}{n^3}$ converges by the p-Test, as it has $p = 3 > 1$, it follows that $\displaystyle\sum_{n=0}^{\infty}\left|\frac{\sin(n)+\cos(n)}{n^3+n^2+n+1}\right|$ converges as well. As this means that the given series converges absolutely, it converges. □

e. We will use the Divergence Test to show that $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{n!}$ diverges. Note that for all $n \geq 1$,

$$\frac{n^n}{n!} = \frac{n}{n}\cdot\frac{n}{n-1}\cdot\frac{n}{n-2}\cdot\cdots\cdot\frac{n}{3}\cdot\frac{n}{2}\cdot\frac{n}{1} \geq n\,,$$

since every factor $\dfrac{n}{k}$ for $k = 2, 3, \ldots, n$ has to be ≥ 1. It follows that

$$\lim_{n\to\infty}\frac{n^n}{n!} \geq \lim_{n\to\infty} n = \infty > 0\,,$$

so the given series must diverge by the Divergence Test. □

Note: The same inequality used above could be used with the Basic Comparison Test to show that the given series diverges by comparison with the series $\sum_{n=1}^{\infty} n$.

f. We will apply the Alternating Series Test to $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$.

i. Since $\sqrt{n+1} > 0$ for all $n \geq 2$ and $(-1)^n$ alternates sign, $\dfrac{(-1)^n}{\sqrt{n+1}}$ alternates sign.

ii. Since $\sqrt{n+2} > \sqrt{n+1}$ for all $n \geq 2$, we have

$$\left|\frac{(-1)^{n+1}}{\sqrt{(n+1)+1}}\right| = \frac{1}{\sqrt{n+2}} < \frac{1}{\sqrt{n+1}} = \left|\frac{(-1)^n}{\sqrt{n+1}}\right|$$

for all $n \geq 2$.

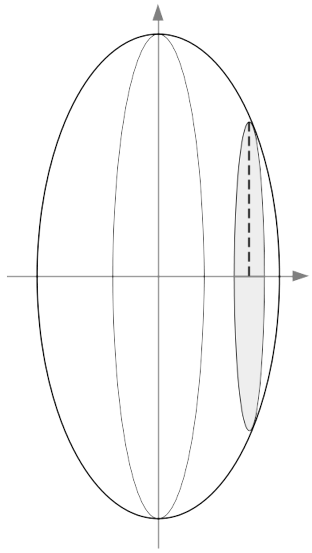
iii. $\displaystyle\lim_{n\to\infty} \left|\frac{(-1)^n}{\sqrt{n+1}}\right| = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} \begin{matrix} \to 1 \\ \to \infty \end{matrix} = 0$

Since it satisfies all three hypotheses of the Alternating Series Test, the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges. ■

Part II. Do any three (3) of **3** – **6**. *[30 = 3×10 each]*

3. Find the volume of the solid obtained by revolving the region below $y = 4 - x^2$ and above $y = 0$, for $-2 \leq x \leq 2$, about the x-axis. *[10]*

SOLUTION. Here is a sketch of the solid:



We'll use the disk-washer method to compute the volume of the solid. Since the axis of revolution is $y = 0$, otherwise known as the the x-axis, the cross-sections of the solid perpendicular to the axis are disks. The disk at x, for $-2 \leq x \leq 2$, has radius $r = y - 0 = 4 - x^2$ and so has area $\pi r^2 = \pi \left(4 - x^2\right)^2$. It follows that the volume is:

$$\begin{aligned}
V &= \int_{-2}^{2} \pi r^2\,dx = \int_{-2}^{2} \pi \left(4 - x^2\right)^2\,dx = \pi \int_{-2}^{2} \left(16 - 8x^2 + x^4\right)\,dx \\
&= \pi \left(16x - 8\frac{x^3}{3} + \frac{x^5}{5} \right) \Big|_{-2}^{2} = \pi \left(16 \cdot 2 - 8\frac{2^3}{3} + \frac{2^5}{5} \right) - \pi \left(16(-2) - 8\frac{(-2)^3}{3} + \frac{(-2)^5}{5} \right) \\
&= \pi \left(32 - \frac{64}{3} + \frac{32}{5} \right) - \pi \left(-32 + \frac{64}{3} - \frac{32}{5} \right) = \pi \left(32 - \frac{64}{3} + \frac{32}{5} + 32 - \frac{64}{3} + \frac{32}{5} \right) \\
&= \pi \left(64 - \frac{128}{3} + \frac{64}{5} \right) = \pi \frac{960 - 640 + 192}{15} = \frac{512}{15}\pi \qquad \blacksquare
\end{aligned}$$

4. **a.** Find the arc-length of the curve $y = \ln(\cos(x))$, where $0 \leq x \leq \frac{\pi}{4}$. *[6]*

b. Find the average value of $\tan(x)$ on the interval $\left[0, \frac{\pi}{4}\right]$. *[4]*

SOLUTIONS. **a.** We'll compute $\frac{dy}{dx}$, plug it into the arc-length formula, and integrate away. First,

$$\frac{dy}{dx} = \frac{d}{dx}\ln(\cos(x)) = \frac{1}{\cos(x)} \cdot \frac{d}{dx}\cos(x) = \frac{1}{\cos(x)} \cdot (-\sin(x)) = -\tan(x)\,.$$

Second, it follows that an increment of arc-length is

$$\begin{aligned} ds &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + (-\tan(x))^2}\, dx \\ &= \sqrt{1 + \tan^2(x)}\, dx = \sqrt{\sec^2(x)}\, dx = \sec(x)\, dx\,. \end{aligned}$$

Third, we compute the arc-length:

$$\begin{aligned} \text{arc-length} &= \int_0^{\pi/4} ds = \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{\pi/4} \sec(x)\, dx \\ &= \ln(\tan(x) + \sec(x))\Big|_0^{\pi/4} = \ln\left(\tan\left(\frac{\pi}{4}\right) + \sec\left(\frac{\pi}{4}\right)\right) - \ln(\tan(0) + \sec(0)) \\ &= \ln\left(1 + \sqrt{2}\right) - \ln(0+1) = \ln\left(1 + \sqrt{2}\right) \qquad \square \end{aligned}$$

b. By definition, the average value of $f(x)$ on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\, dx$. We plug in the given function and interval and integrate:

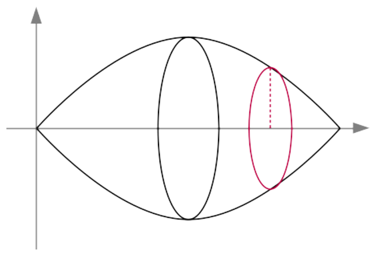
$$\text{Average value} = \frac{1}{\frac{\pi}{4} - 0}\int_0^{\pi/4} \tan(x)\, dx = \frac{4}{\pi}(-\ln(\cos(x)))\Big|_0^{\pi/4}$$

The anti-derivative was obtained by rearranging the derivative in the first part of the solution to **a** above.

$$\begin{aligned} &= \frac{4}{\pi}\left(-\ln\left(\cos\left(\frac{\pi}{4}\right)\right)\right) - \frac{4}{\pi}(-\ln(\cos(0))) = -\frac{4}{\pi}\ln\left(\frac{1}{\sqrt{2}}\right) + \frac{4}{\pi}\ln(1) \\ &= -\frac{4}{\pi}\ln\left(2^{-1/2}\right) + \frac{4}{\pi} \cdot 0 = -\frac{4}{\pi} \cdot \left(-\frac{1}{2}\right)\ln(2) = \frac{2\ln(2)}{\pi} \qquad \blacksquare \end{aligned}$$

5. Find the area of the surface obtained by revolving the curve $y = \sin(x)$, for $0 \le x \le \pi$, about the x-axis. *[10]*

SOLUTION. Here is a crude sketch of the surface:



We will plug the given curve and axis of rotation into the surface area formula and integrate away. First, $\frac{dy}{dx} = \frac{d}{dx}\sin(x) = \cos(x)$, so $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1+\cos^2(x)}\, dx$. Second, the piece of the curve at x is rotated around a circle of radius $r = y - 0 = \sin(x)$. It follows that $SA = \int_0^\pi 2\pi r\, ds = 2\pi \int_0^\pi \sin(x)\sqrt{1+\cos^2(x)}\, dx$. This integral is the same, except for multiplication by 2π and the upper limit of integration, as the integral in **1a** and we do it in the same way. We will use first use the substitution $u = \cos(x)$, so $du = (-1)\sin(x)\, dx$ and $\sin(x)\, dx = (-1)\, du$, and change the limits as we go along: $\begin{matrix} x & 0 & \pi \\ u & 1 & -1 \end{matrix}$.

$$SA = \int_0^\pi 2\pi \sin(x)\sqrt{1+\cos^2(x)}\, dx = 2\pi \int_1^{-1} \sqrt{1+u^2}\,(-1)\, du = 2\pi \int_{-1}^{1} \sqrt{1+u^2}\, du$$

Substitute again, this time with $u = \tan(t)$, so $du = \sec^2(t)\, dt$, and change the limits accordingly: $\begin{matrix} u & -1 & 1 \\ t & -\pi/4 & \pi/4 \end{matrix}$.

$$\begin{aligned}
SA &= 2\pi \int_{-1}^{1} \sqrt{1+u^2}\, du = 2\pi \int_{-\pi/4}^{\pi/4} \sqrt{1+\tan^2(t)}\, \sec^2(t)\, dt = 2\pi \int_{-\pi/4}^{\pi/4} \sqrt{\sec^2(t)}\, \sec^2(t)\, dt \\
&= 2\pi \int_{-\pi/4}^{\pi/4} \sec(t)\sec^2(t)\, dt = \int_{-\pi/4}^{\pi/4} \sec^3(t)\, dt \quad \text{Apply the secant reduction formula:} \\
&= 2\pi \frac{1}{3-1} \tan(t)\sec^{3-2}(t)\Big|_{-\pi/4}^{\pi/4} + 2\pi \frac{3-2}{3-1} \int_{-\pi/4}^{\pi/4} \sec^{3-2}(t)\, dt \\
&= 2\pi \frac{1}{2} \tan(t)\sec(t)\Big|_{-\pi/4}^{\pi/4} + 2\pi \frac{1}{2} \ln\left(\tan(t) + \sec(t)\right)\Big|_{-\pi/4}^{\pi/4} \\
&= \pi \cdot 1 \cdot \sqrt{2} - \pi \cdot (-1) \cdot \sqrt{2} + \pi \ln\left(1+\sqrt{2}\right) - \pi \ln\left(-1+\sqrt{2}\right) \\
&= 2\pi\sqrt{2} + \pi \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) \qquad \blacksquare
\end{aligned}$$

6. Work out $\int \frac{x^3 - x^2 + x + 59}{x^3 - x^2 + x - 1}\, dx$. *[10]*

SOLUTION. The integrand is a rational function, so we haul out the "partial fractions" technology.

i. We first ensure that the degree of the numerator is less than the degree of the denominator. In the given integrand thre degree of both is 3. We can deal with this using long division, but in this case the numerator and denominator are almost the same, so there is a neat shortcut:

$$\begin{aligned}\frac{x^3 - x^2 + x + 59}{x^3 - x^2 + x - 1} &= \frac{x^3 - x^2 + x + -1 + 60}{x^3 - x^2 + x - 1} \\ &= \frac{x^3 - x^2 + x - 1}{x^3 - x^2 + x - 1} + \frac{60}{x^3 - x^2 + x - 1} \\ &= 1 + \frac{60}{x^3 - x^2 + x - 1}\end{aligned}$$

ii. We factor the denominator: $x^3 - x^2 + x - 1 = x^2(x-1) + (x-1) = \left(x^2+1\right)(x-1)$. Since $x^2 + 1 \geq 1 > 0$ for all real x, $x^2 + 1$ is an irreducible quadratic, so we cannot factor the denominator any further.

iii. We decompose the remaining rational function into partial fractions using the factorization of the denominator.

$$\frac{60}{x^3 - x^2 + x - 1} = \frac{60}{\left(x^2+1\right)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1},$$

for some constants A, B, and C such that $(Ax+B)(x-1) + C\left(x^2+1\right) = 60$. (Put the partial fractions over the common denominator $\left(x^2+1\right)(x-1)$ and equate numerators.)

iv. We solve for the constants A, B, and C. Since

$$60 = (Ax+B)(x-1) + C\left(x^2+1\right) = (A+C)x^2 + (-A+B)x + (-B+C),$$

it follows that $A+C=0$, $-A+B=0$, and $-B+C=60$. Adding the three equations together gives us $2C = (A+C) + (-A+B) + (-B+C) = 60$, so $C = 30$. Plugging $C = 30$ into the first equation yields $A = -C = -30$ and plugging this into the second equation yields $B = A = -30$. Thus

$$\frac{60}{x^3 - x^2 + x - 1} = \frac{60}{\left(x^2+1\right)(x-1)} = \frac{-30x-30}{x^2+1} + \frac{30}{x-1}.$$

v. We decompose the given integral using all of the above and proceed to compute it:

$$\begin{aligned}\int \frac{x^3 - x^2 + x + 59}{x^3 - x^2 + x - 1}\, dx &= \int \left[1 + \frac{60}{x^3 - x^2 + x - 1}\right] dx = x + \int \frac{60}{\left(x^2+1\right)(x-1)}\, dx \\ &= x + \int \frac{-30x-30}{x^2+1}\, dx + \int \frac{30}{x-1}\, dx \\ &= x - 30\int \frac{x-1}{x^2+1}\, dx + 30\int \frac{1}{x-1}\, dx \\ &= x - 30\int \frac{x}{x^2+1}\, dx - 30\int \frac{-1}{x^2+1}\, dx + 30\int \frac{1}{x-1}\, dx\end{aligned}$$

In the first of the three remaining integrals we will use the substitution $u = x^2 + 1$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, and in the third of the integrals we will use the substitution $w = x - 1$, so $dw = dx$. Then:

$$\begin{aligned}
\int \frac{x^3 - x^2 + x + 59}{x^3 - x^2 + x - 1}\,dx &= x - 30\int \frac{x}{x^2+1}\,dx - 30\int \frac{-1}{x^2+1}\,dx + 30\int \frac{1}{x-1}\,dx \\
&= x - 30\int \frac{1}{u}\cdot\frac{1}{2}\,du + 30\int \frac{1}{x^2+1}\,dx + 30\int \frac{1}{w}\,dw \\
&= x - 15\ln(u) + 30\arctan(x) + 30\ln(w) + K \\
&= x - 15\ln\left(x^2+1\right) + 30\arctan(x) + 30\ln(x-1) + K \qquad \blacksquare
\end{aligned}$$

Part III. Do any three (3) of **7** – **10**. *[30 = 3×10 each]*

7. Determine the radius and interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{2n}$. What function has this power series as its Taylor series at 0? *[10]*

SOLUTION. As usual, we will lead with the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{\frac{x^{2(n+1)}}{2(n+1)}}{\frac{x^{2n}}{2n}} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+2}}{2n+2} \cdot \frac{2n}{x^{2n}} \right| = \lim_{n\to\infty} \left| x^2 \cdot \frac{n}{n+1} \right| = x^2 \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$
$$= x^2 \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = x^2 \cdot \frac{1}{1+0} = x^2$$

Thus, by the Ratio Test, the series converges absolutely when $x^2 < 1$, *i.e.* $-1 < x < 1$, and diverges when $x^2 > 1$, *i.e.* $x < -1$ or $x > 1$. It follows that the radius of convergence of this series is $r = 1$. When $x^2 = 1$, *i.e.* $x = \pm 1$, the Ratio Test is inconclusive, so we handle these cases separately:

$x = -1$: In this case the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{2n} = \sum_{n=1}^{\infty} \frac{1^n}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges because it is a constant multiple of the harmonic series. (Alternatively, one could apply the p-Test since $p = 1$ here, or use the Integral Test.)

$x = 1$: In this case the series is $\displaystyle\sum_{n=1}^{\infty} \frac{1^{2n}}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges because it is a constant multiple of the harmonic series. (Again, one could use the p-Test or Integral Test.)

Thus the interval of convergence of the given power series is $(-1, 1)$.

It remains to discover what function has the given power series as its Taylor series at 0. $\dfrac{x^{2n}}{2n}$ is the antiderivative of x^{2n-1} for all $n \geq 1$, so the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{2n}$ is the antiderivative of the power series $\displaystyle\sum_{n=1}^{\infty} x^{2n-1} = x + x^3 + x^5 + \cdots$. The last is a geometric series with first term $a = x$ and common ratio $r = x^2$, so it sums to $\dfrac{a}{1-r} = \dfrac{x}{1-x^2}$ when it converges. It follows that the given series sums to the antiderivative of this expression:

$$\int \frac{x}{1-x^2}\,dx = \int \frac{1}{u} \cdot \left(-\frac{1}{2}\right) du \qquad \begin{array}{l}\text{Using the substitution } u = 1 - x^2, \\ \text{so } du = -2x\,dx \text{ and } x\,dx = \left(-\frac{1}{2}\right) du.\end{array}$$
$$= -\frac{1}{2}\ln(u) + C = -\frac{1}{2}\ln\left(1 - x^2\right) + C$$

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{2n} = -\frac{1}{2}\ln\left(1 - x^2\right) + C$ for some constant C when the series converges. Plugging $x = 0$ in on both sides tells us that $0 = 0 + C$, so $C = 0$. Since a function equal to a power series has that series as its Taylor series, it follows that the given series is the Taylor series at 0 of $f(x) = -\frac{1}{2}\ln\left(1 - x^2\right)$. ■

8. Consider the rational function $q(x) = \dfrac{x^7 - 1}{x - 1}$. Find the Taylor series at 0 of $q(x)$ and determine its radius and interval of convergence. *[10]*

SOLUTION. This is more or less a trick question. Note that $x = 1$ is a root of $x^7 - 1$, so $x - 1$ is a factor of $x^7 - 1$. Since

$$\begin{array}{rrrrrrrrrrrrrrrrr}
 & & & & x^6 & + & x^5 & + & x^4 & + & x^3 & + & x^2 & + & x & + & 1 \\
x-1 \;) & & x^7 & + & 0x^6 & + & 0x^5 & + & 0x^4 & + & 0x^3 & + & 0x^2 & + & 0x & - & 1 \\
 & - & [x^7 & - & x^6] & & & & & & & & & & & & \\
 & & & & x^6 & + & 0x^5 & & & & & & & & & & \\
 & & & - & [x^6 & - & x^5] & & & & & & & & & & \\
 & & & & & & x^5 & + & 0x^4 & & & & & & & & \\
 & & & & & - & [x^5 & - & x^4] & & & & & & & & \\
 & & & & & & & & x^4 & + & 0x^3 & & & & & & \\
 & & & & & & & - & [x^4 & - & x^3] & & & & & & \\
 & & & & & & & & & & x^3 & + & 0x^2 & & & & \\
 & & & & & & & & & - & [x^3 & - & x^2] & & & & \\
 & & & & & & & & & & & & x^2 & + & 0x & & \\
 & & & & & & & & & & & - & [x^2 & - & x] & & \\
 & & & & & & & & & & & & & & x & - & 1 \\
 & & & & & & & & & & & & & - & [x & - & 1] \\
 & & & & & & & & & & & & & & & & 0
\end{array}$$

it follows that $x^7 - 1 = (x-1)\left(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1\right)$, and so $q(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$, except at $x = 1$, where the original definition of $q(x)$ does not make sense. (What happens with $q(x)$ at $x = 1$ makes no difference to the definition of the Taylor series of $q(x)$ at 0, though.)

A power series equal to a function is the Taylor series of that function. Every polynomial is a power series in a trivial way, in particular

$$q(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \sum_{n=7}^{\infty} 0x^n$$

(except at 1), so the Taylor series of $q(x)$ is $1 + x + x^2 + x^3 + x^4 + x^5 + x^6$. (Note that this is a somewhat uncommon example of a Taylor series that is defined where the original function is not ...).

Every polynomial is defined for all real numbers, so this Taylor series has radius of convergence is $r = \infty$ and interval of convergence is $(-\infty, \infty)$. ∎

9. Find the Taylor series at 0 of $f(x) = \dfrac{1}{3+x}$ and determine its radius and interval of convergence. *[10]*

SOLUTION. *(Brute force using Taylor's formula.)* We build the usual table to find the values of the derivatives of $f(x) = \dfrac{1}{3+x} = (3+x)^{-1}$ at 0:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(3+x)^{-1}$	$\dfrac{1}{3}$
1	$(-1)(3+x)^{-2}$	$\dfrac{-1}{3^2}$
2	$(-1)^2 2(3+x)^{-3}$	$\dfrac{(-1)^2 2}{3^3}$
3	$(-1)^3 6(x+3)^{-4}$	$\dfrac{(-1)^3 6}{3^4}$
$\vdots$	$\vdots$	$\vdots$
k	$(-1)^k k!(3+x)^{-(k+1)}$	$\dfrac{(-1)^k k!}{3^{k+1}}$
$\vdots$	$\vdots$	$\vdots$

It follows that the Taylor series at 0 of $f(x) = \dfrac{1}{3+x}$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n n!/3^{n+1}}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^{n+1}},$$

which is a geometric series with first term $a = \frac{1}{3}$ and common ratio $r = -\dfrac{x}{3}$. Thus it converges when $|r| = \left|-\dfrac{x}{3}\right| = \dfrac{|x|}{3} < 1$, *i.e.* when $-3 < x < 3$, and diverges otherwise. Hence the radius of convergence of the Taylor series is 3 and the interval of convergence is $(-3, 3)$. □

SOLUTION. *(Cunning and algebra.)* $f(x) = \dfrac{1}{3+x} = (3+x)^{-1}$ looks almost like the sum a geometric series, which generally has the form $\dfrac{a}{1-r}$, where a is the first term and r is the common ratio of the series. A little algebra will put in such a form:

$$f(x) = \frac{1}{3+x} = \frac{1}{3+x} \cdot \frac{\frac{1}{3}}{\frac{1}{3}} = \frac{\frac{1}{3}}{1+\frac{x}{3}} = \frac{\frac{1}{3}}{1-\left(-\frac{x}{3}\right)}$$

It follows that $f(x) = \dfrac{1}{3} - \dfrac{x}{3^2} + \dfrac{x^2}{3^3} - \dfrac{x^3}{3^4} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^{n+1}}$ (when the series converges), so this is the Taylor series at 0 of $f(x)$. Since it is a geometric series with common ratio $r = -\dfrac{x}{3}$, it converges exactly when $|r| = \left|-\dfrac{x}{3}\right| = \dfrac{|x|}{3} < 1$, *i.e.* exactly when $-3 < x < 3$. It follows that the radius of convergence is 3 and the interval of convergence is $(-3, 3)$. ■

10. In each case, give an example (or explain why there isn't one) of a series $\displaystyle\sum_{n=2}^{\infty} a_n$

a. ... that diverges, but $\displaystyle\sum_{n=2}^{\infty} (-1)^n a_n$ converges. *[1]*

b. ... that converges, but $\displaystyle\sum_{n=2}^{\infty} (-1)^n a_n$ diverges. *[1]*

c. ... that diverges, but $\displaystyle\sum_{n=2}^{\infty} a_n^2$ converges. *[2]*

d. ... that converges, but $\displaystyle\sum_{n=2}^{\infty} a_n^2$ diverges. *[2]*

e. ... that converges conditionally, but $\displaystyle\sum_{n=2}^{\infty} (-1)^n a_n$ converges absolutely. *[2]*

f. ... that converges absolutely, but $\displaystyle\sum_{n=2}^{\infty} (-1)^n a_n$ converges conditionally. *[2]*

SOLUTIONS. **a.** The harmonic series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, but the the alternating harmonic series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n}$ converges, both of which facts were worked through in lecture and in the textbook. (Note that starting a series at $n = 2$ instead of $n = 1$ or $n = 0$ does not affect whether it converges or diverges.) □

b. The alternating harmonic series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n}$ converges, but the series $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{(-1)^n}{n} = \sum_{n=2}^{\infty} \frac{(-1)^{2n}}{n} = \sum_{n=2}^{\infty} \frac{1}{n}$ is the harmonic series, which diverges. □

c. The harmonic series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, but $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges by the p-Test, since it has $p = 2 > 1$. □

d. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test:

i. $\sqrt{n} > 0$ when $n \geq 2$ while $(-1)^n$ alternates sign, so $\dfrac{(-1)^n}{\sqrt{n}}$ alternates sign as n increases.

ii. $\displaystyle\left|\frac{(-1)^{n+1}}{\sqrt{n+1}}\right| = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = \left|\frac{(-1)^n}{\sqrt{n}}\right|$ for all $n \geq 2$, since $\sqrt{n+1} > \sqrt{n}$.

iii. $\displaystyle\lim_{n\to\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right| = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, since $\sqrt{n} \to \infty$ as $n \to \infty$.

On the other hand, $\displaystyle\sum_{n=2}^{\infty}\left(\frac{(-1)^n}{\sqrt{n}}\right)^2 = \sum_{n=2}^{\infty}\frac{1}{n}$ is the harmonic series (again!), which diverges. $\square$

e & f. There are no such examples. Since $|a_n| = |(-1)^n a_n|$ for all n, if either of $\displaystyle\sum_{n=2}^{\infty} a_n$ or $\displaystyle\sum_{n=2}^{\infty}(-1)^n a_n$ converges absolutely, *i.e.* $\displaystyle\sum_{n=2}^{\infty}|a_n|$ converges, then so does the other, which means the other does not converge conditionally. ■

[Total = 100]

Part IV. Bonus! If you want to, do one or both of the following problems.

41. Write a poem touching on calculus or mathematics in general. *[1]*

SOLUTION. Here is a haiku by your instructor:

Can you count the words?
If not, then observe
two words: math is hard.

The last line is not actually original, as it's from a long-ago comment on the snarky Saskatchewan political blog *Small Dead Animals* (`www.smalldeadanimals.com`). After a few years I realized that it had the right number of syllables for a line of a haiku. ■

42. When does $6 \times 9 = 42$ actually work? (With apologies to Douglas Adams. :-) *[1]*

SOLUTION. This equation works in base 13:

$$6 \times 9 = 54 = 52 + 2 = 4 \cdot 13^1 + 2 \cdot 13^0 = 42_{13}$$

I still don't know if Adams knew this as he was writing *The Hitchhiker's Guide to the Galaxy*. It was certainly pointed out after publication! ■