Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Summer 2018

Solutions to Assignments #3 & 4
Optimal Cone & Optimal Cone: The Sequel

Recall the sole question from Assignment #3:

1. A right circular cone with radius r and height h has volume $V = \frac{1}{3}\pi r^2 h$ and surface area (counting the area of the circle at the non-pointy end) of $A = \pi r^2 + \pi r\sqrt{r^2 + h^2}$. Suppose that such a cone is to have a total volume of 100 L. What is the minimum possible surface area of such a cone? *[10]*

It's a pretty good bet that if you tried to do this, things got just a bit messy. This time you get to have `Maple` do much of the work. `Maple` has several operations and commands that might be helpful. In particular, the `diff` operator takes the derivative of an expression and the `solve` command and its relatives, especially `fsolve`, are often useful if you need to solve an equation. Please read up on the basics of these and other possibly useful commands in Prof. Urroz's introductions to using Maple [1] and [2].

1. *(The sequel.)* Answer question **1** from Assignment #3, using `Maple` as much as possible to perform the actual symbolic manipulations and computations. Please include the printout(s) of your `Maple` work with your solution. *[10]*

NOTE. You may use other sofware, such as `Mathematica` or `SageMath`, with similar capabilities instead of `Maple` if you wish.

SOLUTION TO **A#3**. *(Mostly by hand.)* Since we are given $V = \dfrac{1}{3}\pi r^2 h = 100$, it follows that $h = \dfrac{100}{\frac{1}{3}\pi r^2} = \dfrac{300}{\pi r^2}$. This lets us express the surface area of the cone in terms of r only:

$$A = \pi r^2 + \pi r\sqrt{r^2 + h^2} = \pi r^2 + \pi r\sqrt{r^2 + \left(\frac{300}{\pi r^2}\right)^2} = \pi r^2 + \sqrt{\pi^2 r^4 + \frac{300^2}{r^2}}$$

For the last step, which was done to make some of the later algebra and calculus a little more convenient, note that when the πr is brought inside the square root, it must be squared.

Note also that r and h must both be positive, but one can make either arbitrarily small at the cost of making the other arbitrarily large, so $0 < r < \infty$. It is not hard to see that $\lim_{r\to 0^+}\left(\pi r^2 + \sqrt{\pi^2 r^4 + \frac{300^2}{r^2}}\right) = \infty$ and $\lim_{r\to\infty}\left(\pi r^2 + \sqrt{\pi^2 r^4 + \frac{300^2}{r^2}}\right) = \infty$, since $\dfrac{300^2}{r^2} \to \infty$ as $r \to 0^+$ and $\pi r^2 \to \infty$ as $r \to \infty$. It follows that if we find a single critical point with $0 < r < \infty$ (as we will!), then it will have to be a minimum, as desired.

On to the actual calculus. With a bit of help from the Power and Chain Rules:

$$\begin{aligned}
\frac{dA}{dr} &= \frac{d}{dr}\left(\pi r^2 + \sqrt{\pi^2 r^4 + \frac{300^2}{r^2}}\right) = \frac{d}{dr}\left(\pi r^2\right) + \frac{d}{dr}\sqrt{\pi^2 r^4 + \frac{300^2}{r^2}} \\
&= 2\pi r + \frac{1}{2\sqrt{\pi^2 r^4 + \frac{300^2}{r^2}}} \cdot \frac{d}{dr}\left(\pi^2 r^4 + \frac{300^2}{r^2}\right) \\
&= 2\pi r + \frac{4\pi^2 r^3 - 2\frac{300^2}{r^3}}{2\sqrt{\pi^2 r^4 + \frac{300^2}{r^2}}} = 2\pi r + \frac{2\pi^2 r^3 - \frac{300^2}{r^3}}{\sqrt{\pi^2 r^4 + \frac{300^2}{r^2}}}
\end{aligned}$$

Now we have to solve for the value(s) of r that make $\frac{dA}{dr} = 0$. Sartre was wrong: Hell isn't other people, it's too much algebra ...

$$\begin{aligned}
\frac{dA}{dr} = 0 &\iff 2\pi r + \frac{2\pi^2 r^3 - \frac{300^2}{r^3}}{\sqrt{\pi^2 r^4 + \frac{300^2}{r^2}}} = 0 \\
&\iff 2\pi r\sqrt{\pi^2 r^4 + \frac{300^2}{r^2}} + 2\pi^2 r^3 - \frac{300^2}{r^3} = 0 \\
&\iff 2\pi r\sqrt{\pi^2 r^4 + \frac{300^2}{r^2}} = -2\pi^2 r^3 + \frac{300^2}{r^3} \\
&\iff 2\pi\sqrt{\pi^2 r^6 + 300^2} = -2\pi^2 r^3 + \frac{300^2}{r^3} \\
&\iff 2\pi r^3\sqrt{\pi^2 r^6 + 300^2} = -2\pi^2 r^6 + 300^2 \\
&\iff 4\pi^2 r^6\left(\pi^2 r^6 + 300^2\right) = \left(-2\pi^2 r^6\right)^2 + 2\left(-2\pi^2 r^6\right)300^2 + \left(300^2\right)^2 \\
&\iff 4\pi^4 r^{12} + 4\pi^2 300^2 r^6 = 4\pi^4 r^{12} - 4\pi^2 300^2 r^6 + 300^4 \\
&\iff 8\pi^2 300^2 r^6 = 300^4 \\
&\iff r^6 = \frac{300^4}{8\pi^2 300^2} = \frac{300^2}{8\pi^2} \\
&\iff r = \left(\frac{300^2}{8\pi^2}\right)^{1/6} \qquad \text{(Since we know } r > 0\text{, we can ignore the negative root.)}
\end{aligned}$$

A little work with a suitable calculator [the not-by-hand part of this solution] now gives that $r \approx 3.232$ (so $h \approx 9.142$) and that the surface area of the conical tank at this value of r is $A \approx 131.3$. The units are fun, sort of: since 1 L is the volume of a cube that is 10 cm = 0.1 m = 1 dm on a side, the native units of length here are decimetres, so the area number is in square decimetres, *i.e.* $A \approx 131.3\ dm^2 = 1.313\ m^3 = 13130\ cm^2$. As was noted earlier, this value must be (approximately) the minimum value, because there is only one critical point with $0 < r < \infty$. Whew! □

SOLUTION TO **A#4**. While one can readily have `Maple` solve for h in terms of r and substitute that into the expression for area, that was easy enough to do by hand that I really couldn't be bothered. Having got $A = \pi r^2 + \sqrt{\pi^2 r^4 + \frac{300^2}{r^2}}$ by hand, it was pretty easy to have `Maple` find the critical point and compute the corresponding area:

```
> fsolve( diff( Pi*r^2 + sqrt( Pi^2 * r^4 + 300^2 / r^2 ), r ) = 0, r )
```

$$3.232030592$$

```
> r := 3.232030592
```

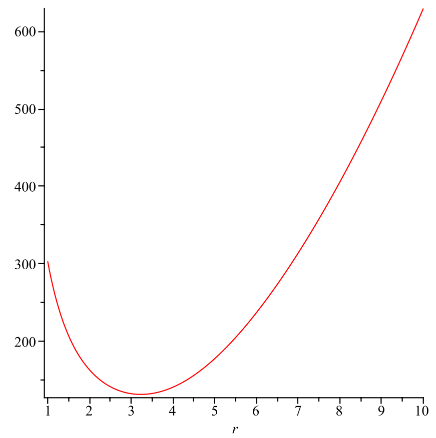
$$r := 3.232030592$$

```
> evalf( Pi*r^2 + sqrt( Pi^2 * r^4 + 300^2 / r^2 ) )
```

$$131.2685808$$

Notice that one never sees what $\frac{dA}{dr}$ actually is! :-) To check that this is indeed a minimum value, the quick and dirty method is to plot the area function:

```
> plot( Pi*r^2 + sqrt( Pi^2 * r^4 + 300^2 / r^2 ), r = 1..10 )
```



Looks like it's a minimum to me! ■

References

1. *Getting started with Maple 10*, by Gilberto E. Urroz (2005), which can found (pdf) at: ~~www.trentu.ca/mathematics/sb/1110H/Summer-2018/GettingStartedMaple10.pdf~~
   euclid.trentu.ca/math/sb/1110H/Summer-2018/GettingStartedMaple10.pdf
2. *A survey of mathematical applications using Maple 10*, by Gilberto E. Urroz (2005), which can found pdf & `Maple` worksheet) at:
   ~~www.trentu.ca/mathematics/sb/1110H/Summer-2018/MathematicsSurveyMaple10.pdf~~
   euclid.trentu.ca/math/sb/1110H/Summer-2018/MathematicsSurveyMaple10.pdf
   or ~~www.trentu.ca/mathematics/sb/1110H/Summer-2018/MathSurveyMaple10.mw~~
   euclid.trentu.ca/math/sb/1110H/Summer-2018/MathematicsSurveyMaple10.mw