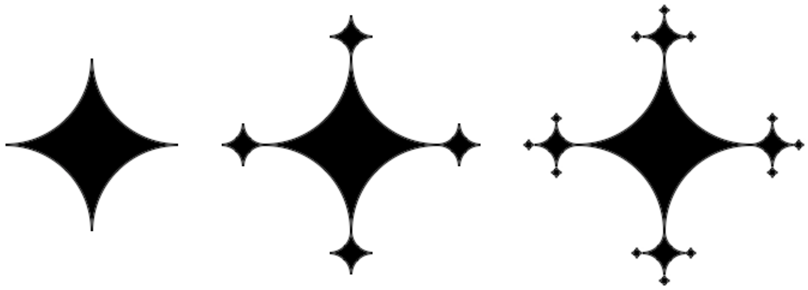
Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Summer 2018

Solutions to Assignment #1
Square - Circle = Squircle?

Let's call the shape that you get by removing four mutually tangent quarter-circles with radius $\frac{s}{2}$ from a square with side length s a *squircle** *of side* s. (See the leftmost shape in the diagram below.)



1. What are the area and perimeter of a squircle of side s? *[1]*

SOLUTION. The perimeter of a squircle of side s is just a rearrangement of four quarters of of the perimeter of a circle of radius $\frac{s}{2}$ from their original circle. It follows that the perimeter of a squircle of side s is just the circumference of a circle of radius $\frac{s}{2}$, namely $2\pi\frac{s}{2} = s\pi$.

The area of a squircle of side s is the area of a square of side s minus the combined areas of four quarters of a circle of radius $\frac{s}{2}$. It follows that the area of a squircle of radius s is $s^2 - \pi\left(\frac{s}{2}\right)^2 = \left(1 - \frac{\pi}{4}\right)s^2$. $\square$

A single squircle has four points where the quarter-circles that were removed met. Consider the following process:

At step $n = 0$ we have a single squircle for which $s = 2$.

At step $n = 1$, we attach four squircles for which $s = \frac{1}{4} \cdot 2 = \frac{1}{2}$ to the squircle in step 0, attaching one (at one of its points) to each point of the larger squircle. (See the middle shape in the diagram above.) The resulting shape has $3 \cdot 4 = 12$ points (where quarter-circles met) to which nothing is yet attached. Let's call these the *free* points of the shape.

At step $n = 2$, we attach a squircle for which $s = \frac{1}{4} \cdot \frac{1}{2} = \left(\frac{1}{4}\right)^2 \cdot 2 = \frac{1}{8}$ to each of the free points in the shape in step 1. (See the rightmost shape in the diagram above.) The resulting shape has $3 \cdot 12 = 3 \cdot (3 \cdot 4) = 3^2 \cdot 4 = 36$ free points.

At step $n = 3$, we attach a squircle for which $s = \frac{1}{4} \cdot \frac{1}{8} = \left(\frac{1}{4}\right)^3 \cdot 2 = \frac{1}{32}$ to each of these the free points in the shape in step 2. (Draw your own picture!) The resulting shape has $3 \cdot 36 = 3 \cdot \left(3^2 \cdot 4\right) = 3^3 \cdot 4 = 108$ free points.

Repeat for each integer $n > 3$ ...

* This shape probably has a name already, but I still don't know it ... :-) It's not an astroid, as it turns out, though it superficially looks like one.

2. Find formulas for the values of s for the squircles added at step n and for the number of free points of the shape obtained in step n. *[2]*

SOLUTION. The value of s used for the squircles at each step is $\frac{1}{4}$ of the previous value. If we refer to the value of s used at step n by s_n, then we have $s_0 = 2 = 2\left(\frac{1}{4}\right)^0$, $s_1 = \frac{1}{4}s_0 = \frac{1}{4}\cdot 2\left(\frac{1}{4}\right)^0 = 2\left(\frac{1}{4}\right)^1$, $s_2 = \frac{1}{4}s_1 = \frac{1}{4}\cdot 2\left(\frac{1}{4}\right)^1 = 2\left(\frac{1}{4}\right)^2$, $s_3 = \frac{1}{4}s_2 = \frac{1}{4}\cdot 2\left(\frac{1}{4}\right)^2 = 2\left(\frac{1}{4}\right)^3$, and so on. The pattern here is pretty clear: $s_n = 2\left(\frac{1}{4}\right)^n$ for each integer $n \geq 0$.

The number of free points at step 0 is obviously 4. At each step a smaller squircle is attached by one of its corners to each free point of the shape from the previous step, leaving the other three points of each new smaller squircle free. This means that the number of free points increases by a factor of 3 from each step to the next. If we refer to the number of free points at step n by f_n, then we have $f_0 = 4 = 4\cdot 3^0$. $f_1 = 3f_0 = 3\cdot 4\cdot 3^0 = 4\cdot 3^1$, $f_2 = 3f_1 = 3\cdot 4\cdot 3^1 = 4\cdot 3^2$, $f_3 = 3f_2 = 3\cdot 4\cdot 3^2 = 4\cdot 3^3$, and so on. Again, the pattern is pretty clear: $f_n = 4\cdot 3^n$ for each integer $n \geq 0$. $\square$

3. Find a formula for the total length of the perimeter of the shape obtained in step n. *[2]*

SOLUTION. Denote the total length of the perimeter of the shape obtained at step n by p_n. Our initial shape is a squircle of side $s = 2$, so, by the answer to question 1, it has perimeter $p_0 = \pi s = 2\pi$. At step 1 we add $f_0 = 4\cdot 3^0$ squircles, each of side $s_1 = 2\left(\frac{1}{4}\right)^1$ and hence each with perimeter $\pi s_1 = 2\pi\left(\frac{1}{4}\right)^1$, for a total perimeter of $p_1 = p_0 + f_0\cdot \pi s_1 = 2\pi + 4\cdot 2\pi\left(\frac{1}{4}\right)^1$. (Note that since the small squircles are attached to the original one at single points, their perimeters do not really overlap, so the their lengths add up without cancellation.) At step 2 we add $f_1 = 4\cdot 3^1$ new squircles, each of side $s_2 = 2\left(\frac{1}{4}\right)^2$ and hence each with perimeter $\pi s_2 = 2\pi\left(\frac{1}{4}\right)^2$, for a total perimeter of $p_2 = p_1 + f_1\cdot\pi s_2 = 2\pi + 4\cdot 3^0\cdot 2\pi\left(\frac{1}{4}\right)^1 + 4\cdot 3^1\cdot 2\pi\left(\frac{1}{4}\right)^2$. Continuing this process we see that at step $n \geq 1$ we get a total perimeter of:

$$\begin{aligned}
p_n &= 2\pi + 4\cdot 3^0\cdot 2\pi\left(\frac{1}{4}\right)^1 + 4\cdot 3^1\cdot 2\pi\left(\frac{1}{4}\right)^2 + \cdots + 4\cdot 3^{n-1}\cdot 2\pi\left(\frac{1}{4}\right)^n \\
&= 2\pi + 2\pi\left[4\cdot 3^0\cdot\left(\frac{1}{4}\right)^1 + 4\cdot 3^1\cdot\left(\frac{1}{4}\right)^2 + \cdots + 4\cdot 3^{n-1}\cdot\left(\frac{1}{4}\right)^n\right] \\
&= 2\pi + 2\pi\left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots + \left(\frac{3}{4}\right)^{n-1}\right] = 2\pi + 2\pi\left[\frac{1-\left(\frac{3}{4}\right)^n}{1-\frac{3}{4}}\right] \\
&= 2\pi + 2\pi\left[\frac{1-\left(\frac{3}{4}\right)^n}{\frac{1}{4}}\right] = 2\pi + 2\pi\cdot 4\left[1-\left(\frac{3}{4}\right)^n\right] = 2\pi + 8\pi\left[1-\left(\frac{3}{4}\right)^n\right] \quad \square
\end{aligned}$$

NOTE. At the key step we used the formula for the sum of a finite geometric series, namely that $a + ar + ar^2 + \cdots + ar^k = a\dfrac{1-r^{k+1}}{1-r}$, which formula works for all real numbers a, all real numbers $r \neq 1$, and all integers $k \geq 0$.

4. Find a formula for the total area of the shape obtained in step n. *[2]*

SOLUTION. Denote the total length of the perimeter of the shape obtained at step n by a_n. Our initial shape is a squircle of side $s = 2$, so, by the answer to question 1, it has area $a_0 = \left(1 - \frac{\pi}{4}\right) s_0^2 = \left(1 - \frac{\pi}{4}\right) 2^2 = 4\left(1 - \frac{\pi}{4}\right) = 4 - \pi$. At step 1 we add $f_0 = 4 \cdot 3^0$ squircles, each of side $s_1 = 2\left(\frac{1}{4}\right)^1$ and hence each with area $\left(1 - \frac{\pi}{4}\right) s_1^2 = \left(1 - \frac{\pi}{4}\right)\left[2\left(\frac{1}{4}\right)^1\right]^2 = 4\left(1 - \frac{\pi}{4}\right)\left(\frac{1}{4}\right)^2 = 4\left(1 - \frac{\pi}{4}\right)\frac{1}{4^2} = (4 - \pi)\frac{1}{4^2}$, for a total area of $a_1 = a_0 + f_0 \cdot \left(1 - \frac{\pi}{4}\right) s_1^2 = (4 - \pi) + 4(4 - \pi)\frac{1}{4^2}$. (Note that since the small squircles are attached to the original one at single points, their areas also do not really overlap, so the their areas add up without cancellation.) At step 2 we add $f_1 = 4 \cdot 3^1$ new squircles, each of side $s_2 = 2\left(\frac{1}{4}\right)^2$ and hence each with area $\left(1 - \frac{\pi}{4}\right) s_2^2 = \left(1 - \frac{\pi}{4}\right)\left[2\left(\frac{1}{4}\right)^2\right]^2 = 4\left(1 - \frac{\pi}{4}\right)\frac{1}{4^4} = (4 - \pi)\frac{1}{4^4}$, for a total area of $a_2 = a_1 + f_1\left(1 - \frac{\pi}{4}\right) S_2^2 = (4 - \pi) + 4(4 - \pi)\frac{1}{4^2} + 4 \cdot 3^1(4 - \pi)\frac{1}{4^4}$. Continuing this process we see that at step $n \geq 1$ we get a total area of:

$$
\begin{aligned}
a_n &= (4 - \pi) + 4(4 - \pi)\frac{1}{4^2} + 4 \cdot 3^1(4 - \pi)\frac{1}{4^4} + \cdots + 4 \cdot 3^{n-1}(4 - \pi)\frac{1}{4^{2n}} \\
&= (4 - \pi)\left[1 + \frac{4}{4^2} + \frac{4 \cdot 3^1}{4^4} + \cdots + \frac{4 \cdot 3^{n-1}}{4^{2n}}\right] \\
&= (4 - \pi)\left[1 + \frac{4}{4^2}\left(1 + \frac{3}{4^2} + \cdots + \frac{3^{n-1}}{4^{2n-2}}\right)\right] \\
&= (4 - \pi)\left[1 + \frac{1}{4}\left(1 + \frac{3}{4^2} + \cdots + \left(\frac{3}{4^2}\right)^{n-1}\right)\right] \\
&= (4 - \pi)\left[1 + \frac{1}{4}\left(1 + \frac{3}{16} + \cdots + \left(\frac{3}{16}\right)^{n-1}\right)\right] \\
&= (4 - \pi)\left[1 + \frac{1}{4}\left(\frac{1 - \left(\frac{3}{16}\right)^n}{1 - \frac{3}{16}}\right)\right] = (4 - \pi)\left[1 + \frac{1}{4}\left(\frac{1 - \left(\frac{3}{16}\right)^n}{\frac{13}{16}}\right)\right] \\
&= (4 - \pi)\left[1 + \frac{1}{4} \cdot \frac{16}{13}\left(1 - \left(\frac{3}{16}\right)^n\right)\right] = (4 - \pi)\left[1 + \frac{4}{13}\left(1 - \left(\frac{3}{16}\right)^n\right)\right] \quad \square
\end{aligned}
$$

5. What are the total length of the perimeter and the total area of the shape obtained after infinitely many steps of the process? *[3]*

SOLUTION. When n gets very large, both $\left(\frac{3}{4}\right)^n$ and $\left(\frac{3}{16}\right)^n$ get very small; you can get them as close as you like to 0 by making n large enough. It follows that the total length of the perimeter and the total area of the shape obtained after infinitely many steps of the process are, respectively:

$$
\begin{aligned}
p &= \lim_{n \to \infty} p_n = 2\pi + 8\pi[1 - 0] = 2\pi + 8\pi = 10\pi \\
a &= \lim_{n \to \infty} a_n = (4 - \pi)\left[1 + \frac{4}{13}(1 - 0)\right] = (4 - \pi)\left(\frac{13}{13} + \frac{4}{13}\right) = \frac{17}{13}(4 - \pi) \quad \blacksquare
\end{aligned}
$$