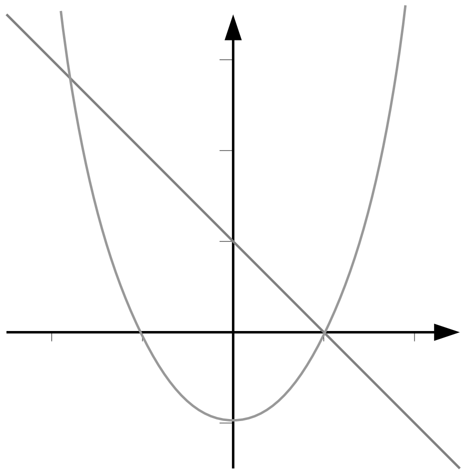
Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Summer 2018

Solutions to the Quizzes

Quiz #1. Wednesday, 9 May. *[10 minutes]*

1. Sketch the line $y = -x + 1$ and the parabola $y = x^2 - 1$. *[2]*
2. Find the coordinates of the points at which the line and the parabola intersect. *[3]*

SOLUTIONS. 1. Here is a crude sketch of the line and the parabola:



□

2. We need to find the points (x, y) which satisfy the equations of both the line and the parabola. At any such point, we would have $-x + 1 = y = x^2 - 1$. Rearranging $-x+1 = x^2-1$ gives $x^2+x-2 = 0$, which we can find the solutions to using the quadratic formula:

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-2)}}{2 \cdot 1} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2} = \begin{cases} \frac{2}{2} = 1 \\ -\frac{4}{2} = -2 \end{cases}$$

When $x = 1$, $y = -x + 1 = -1 + 1 = 0$, and when $x = -2$, $y = -x + 1 = -(-2) + 1 = 2 + 1 = 3$. The coordinates of the ~~pints~~ points at which the line and the parabola intersect are therefore $(1, 0)$ and $(-2, 3)$. ■

Quiz #2. Monday, 14 May. *[10 minutes]*

Use the rules for manipulating limits to compute both of the following:

1. $\lim_{x \to 2} \frac{x-1}{x+1}$ *[2.5]* 2. $\lim_{x \to 2} \frac{x^2 - 3x + 2}{x^2 - x - 2}$ *[2.5]* [*Hint:* The two limits are equal.]

SOLUTIONS. 1. $\lim_{x \to 2} \frac{x-1}{x+1} = \frac{\lim_{x \to 2}(x-1)}{\lim_{x \to 2}(x+1)} = \frac{\left(\lim_{x \to 2} x\right) - \left(\lim_{x \to 2} 1\right)}{\left(\lim_{x \to 2} x\right) + \left(\lim_{x \to 2} 1\right)} = \frac{2-1}{2+1} = \frac{1}{3}$ □

2. $\lim_{x \to 2} \frac{x^2 - 3x + 2}{x^2 - x - 2} = \lim_{x \to 2} \frac{(x-1)(x-2)}{(x+1)(x-2)} = \lim_{x \to 2} \frac{x-1}{x+1} = \frac{1}{3}$, using the solution to 1. ■

Quiz #3. Wednesday, 16 May. *[10 minutes]*

Compute the derivatives of each of the following: 1. $p(x) = 3x^2 - 4x + \pi$ *[1]*

2. $h(x) = xe^{-x}$ *[1]* 3. $g(x) = \dfrac{x^2-1}{x^2+1}$ *[1.5]* 4. $f(x) = \cos^2\left(x^3\right)$ *[1.5]*

SOLUTIONS. 1. The main tool here is the Power Rule:

$$p'(x) = \frac{d}{dx}\left(3x^2 - 4x + \pi\right) = 3\frac{d}{dx}x^2 - 4\frac{d}{dx}x + \frac{d}{dx}\pi = 3\cdot 2x - 4\cdot 1 + 0 = 6x - 4 \quad \square$$

2. The main tools here are the Product Rule and the Chain Rule:

$$\begin{aligned} h'(x) &= \frac{d}{dx}\left(xe^{-x}\right) = \left(\frac{d}{dx}x\right)\cdot e^{-x} + x\cdot\left(\frac{d}{dx}e^{-x}\right) = 1\cdot e^{-x} + x\cdot e^{-x}\cdot\left(\frac{d}{dx}(-x)\right) \\ &= e^{-x} + xe^{-x}\cdot(-1) = (1-x)e^{-x} \quad \square \end{aligned}$$

3. The main tools here are the Power Rule and the Quotient Rule:

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) = \frac{\left(\frac{d}{dx}\left(x^2-1\right)\right)\left(x^2+1\right) - \left(x^2-1\right)\left(\frac{d}{dx}\left(x^2+1\right)\right)}{\left(x^2+1\right)^2} \\ &= \frac{(2x-0)\left(x^2+1\right) - \left(x^2-1\right)(2x+0)}{\left(x^2+1\right)^2} = \frac{\left(2x^3+2x\right) - \left(2x^3-2x\right)}{\left(x^2+1\right)^2} \\ &= \frac{2x^3+2x-2x^3-(-2x)}{\left(x^2+1\right)^2} = \frac{4x}{\left(x^2+1\right)^2} \quad \square \end{aligned}$$

4. The main tools here are the Power Rule and the Chain Rule:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\cos^2\left(x^3\right) = 2\cos\left(x^3\right)\cdot\frac{d}{dx}\cos\left(x^3\right) = 2\cos\left(x^3\right)\cdot\left(-\sin\left(x^3\right)\right)\cdot\frac{d}{dx}x^3 \\ &= -2\cos\left(x^3\right)\sin\left(x^3\right)\cdot 3x^2 = -6x^2\cos\left(x^3\right)\sin\left(x^3\right) \end{aligned}$$

The trigonometrically obsessed may rewrite this answer in any number of ways; for one nice one, as $-3x^2\sin\left(2x^3\right)$. ■

Quiz #4. Wednesday, 23 May. *[10 minutes]*

1. Find the domain and any and all intercepts, intervals of increase and decrease, and maximum and minimum points of $f(x) = x^3 - 3x$. *[5]*

SOLUTIONS. *i. Domain.* $f(x) = x^3 - 3x$ makes sense for every real value of x, so the domain of $f(x)$ is $\mathbb{R} = (-\infty, \infty)$.

ii. Intercepts. For the y-intercept(s), we set $x = 0$. Then $y = f(0) = 0^3 - 3 \cdot 0 = 0$, so the y-intercept is 0. (Note that this means that the point $(0, 0)$ is also an x-intercept.)

For the x-intercept(s), we set $y = x^3 - 3x = 0$ and solve for x. Since

$$y = x^3 - 3x = x\left(x^2 - 3\right) = x\left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right),$$

we can see that $y = 0$ exactly when $x = 0$, $x = -\sqrt{3}$, or $x = \sqrt{3}$, so these are the x-intercepts.

iii. Intervals of increase and decrease. We first find the derivative of $f(x)$:

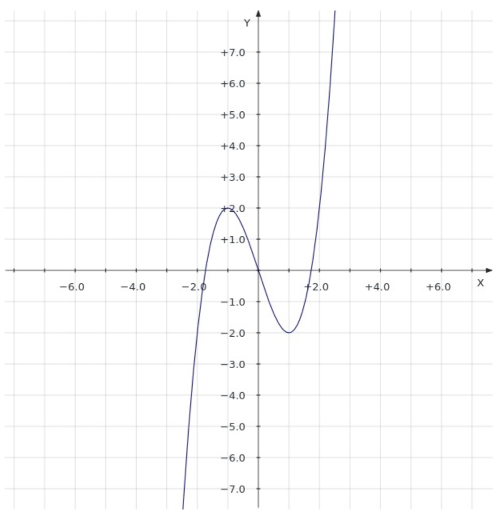
$$f'(x) = \frac{d}{dx}\left(x^3 - 3x\right) = 3x^2 - 3 = 3\left(x^2 - 1\right) = 3(x+1)(x-1)$$

Note that the derivative is defined for all x and is 0 exactly when $x = -1$ or $x = 1$. When $x < -1$, both $x + 1$ and $x - 1$ are negative, so $f'(x) = 3(x+1)(x-1)$ must be positive and hence $f(x)$ be increasing; when $-1 < x < 1$, $x + 1$ is positive and $x - 1$ is negative, so $f'(x) = 3(x+1)(x-1)$ must be negative and hence $f(x)$ be decreasing; and when $x > 1$, both $x + 1$ and $x - 1$ are positive, so $f'(x) = 3(x+1)(x-1)$ must be positive and hence $f(x)$ be increasing. As usual, we summarize this in a table:

x	$(-\infty, -1)$	-1	$(-1, 1)$	1	$(1, \infty)$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\uparrow$	max	$\downarrow$	min	$\uparrow$

iv. Maximum and minimum points. From the table above, $f(x)$ has a (local) maximum at $x = -1$ and a (local) minimum at $x = 1$. Since $f(-1) = (-1)^3 - 3(-1) = 2$ and $f(1) = 1^3 - 3 \cdot 1 = -2$, the corresponding (local) minimum and maximum points are $(-1, 2)$ and $(-1, 2)$, respectively.

v. The graph. Cheated a bit and used a computer to draw it:



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Quiz #5. Wednesday, 30 May. *[10 minutes]*

1. What is the least possible sum of two positive numbers u and w that have a product of 16? *[5]*

SOLUTION. We need to minimize the sum $u + w$ with the constraints that $u > 0$, $w > 0$, and that $uw = 16$. Since we must have $uw = 16$ and $w > 0$, we can let $u = \frac{16}{w}$. Note that w can be any positive real number, so our task is to minimize $f(w) = u + w = \frac{16}{w} + w$ for $w \in (0, \infty)$.

First, we find any critical point(s) of $f(w)$ in $(0, \infty)$.

$$f'(w) = \frac{d}{dw}\left(\frac{16}{w} + w\right) = -\frac{16}{w^2} + 1,$$

which $= 0$ exactly when $-\frac{16}{w^2} = -1$, *i.e.* when $w^2 = 16$. Since $w > 0$, it follows that $f'(w) = 0$ when $w = +\sqrt{16} = 4$.

Second, we check what happens at the endpoints of the interval $(0, \infty)$. Since the endpoints are not actually possible values of w, we take limits:

$$\lim_{w \to 0^+} f(w) = \lim_{w \to 0^+} \left(\frac{16}{w} + w\right) = +\infty + 0 = +\infty$$
$$\lim_{w \to +\infty} f(w) = \lim_{w \to +\infty} \left(\frac{16}{w} + w\right) = 0 + \infty = +\infty$$

Comparing the value of $f(w)$ at the critical point $w = 4$, $f(4) = \frac{16}{4} + 4 = 4 + 4 = 8$, with the values as we approach the endpoints of the interval $(0, \infty)$, we see that $f(w)$ has a minimum value of 8 for $0 < w < \infty$. It follows that 8 is the least possible sum of two positive numbers whose product is equal to 16. ■

Quiz #6. Monday, 4 June. *[10 minutes]*

1. A tank shaped like a rectangular box measures $1\ m \times 1\ m$ at the base and is $2\ m$ high. It is initially empty, but then water is poured into the tank at the rate of $50\ L/m$, and no water is allowed to drain or leak while this is done. How is the level of water in the tank changing at the instant that water in the tank is $1\ m$ deep? *[10 minutes]*

SOLUTION. *Interpreting L/m as litres per minute.* Let h be the depth of the water in the tank in metres. The water then occupies a volume in the shape of a rectangular box which measures $1\ m \times 1\ m$ at the base and is $h\ m$ high, so the volume is $V = 1 \cdot 1 \cdot h = h\ m^3$. Since $1\ L = 0.001\ m^3$, we have that $\frac{dV}{dt} = 50\ L/min = 0.05\ m^3/min$. On the other hand, $\frac{dV}{dt} = \frac{d}{dt}h = \frac{dh}{dt}$, so $\frac{dh}{dt} = 0.05\ m/min$. [Note that the level of water in the tank turns out to be irrelevant here.] □

SOLUTION. *Interpreting L/m as litres per metre, so inflow is proportional to the level.* Things work in much the same way at the given instant and give the same number, because the specified instant is when $h = 1\ m$, so $\frac{dV}{dh} = 50h\ L/m$ works out to 0.05 too. □

Quiz #7. Wednesday, 6 June. *[12 minutes]*

Compute each of the following integrals: 1. $\displaystyle\int_0^{\pi} \sin(x)\,dx$ *[1]* 2. $\displaystyle\int \left(x^2 + \pi e^x\right)\,dx$ *[1]*

3. $\displaystyle\int_2^4 (2x+1)\,dx$ *[1]* 4. $\displaystyle\int \sec^2(x)\,dx$ *[1]* 5. $\displaystyle\int_1^e \frac{1}{x}\,dx$ *[1]*

SOLUTIONS. 1. $\frac{d}{dx}\cos(x) = -\sin(x)$, so $\frac{d}{dx}(-\cos(x)) = -(-\sin(x)) = \sin(x)$, so the antiderivative of $\sin(x)$ is $-\cos(x) + C$. Hence

$$\int_0^{\pi} \sin(x)\,dx = -\cos(x)\Big|_0^{\pi} = [-\cos(\pi)] - [-\cos(0)] = [-(-1)] - [-1] = 1 + 1 = 2\,. \quad \square$$

2. Using the basic properties of integrals, the Power Rule for integration, and the fact that e^x is its own derivative and antiderivative,

$$\int \left(x^2 + \pi e^x\right)\,dx = \int x^2\,dx + \pi \int e^x\,dx = \frac{x^{2+1}}{2+1} + \pi e^x + C = \frac{x^3}{3} + \pi e^x + C\,.$$

Note that since this is an indefinite integral, we need to include the generic constant C. $\square$

3. Using the basic properties of integrals and the Power Rule for integration, we have:

$$\begin{aligned}\int_2^4 (2x+1)\,dx &= 2\int_2^4 x\,dx + \int_2^4 1\,dx = 2 \cdot \left.\frac{x^{1+1}}{1+1}\right|_2^4 + x\Big|_2^4 = x^2\Big|_2^4 + [4-2] \\ &= \left[4^2 - 2^2\right] + 2 = [16-4] + 2 = 12 + 2 = 14 \quad \square\end{aligned}$$

4. Since $\frac{d}{dx}\tan(x) = \sec^2(x)$, we have that $\int \sec^2(x)\,dx = \tan(x) + C$. $\square$

5. Since $\frac{d}{dx}\ln(x) = \frac{1}{x}$, or by using the $n = -1$ exception to the Power Rule for integration, we have $\int_1^e \frac{1}{x}\,dx = \ln(x)\Big|_1^e = \ln(e) - \ln(1) = \ln\left(e^1\right) - \ln\left(e^0\right) = 1 - 0 = 0.$ ■

Quiz #8. Monday, 11 June. *[10 minutes]*

1. Compute $\displaystyle\int_{-1/2}^{(\pi-4)/8} \frac{e^{\tan(2x+1)}}{\cos^2(2x+1)}\,dx$. *[5]* NOTE: $\cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

SOLUTION. First, we'll use the substitution $u = 2x+1$, so $\frac{du}{dx} = \frac{d}{dx}(2x+1) = 2$, and thus $du = 2\,dx$, from which we get that $dx = \frac{1}{2}\,du$. We will also change the limits as we go along. Note that $2\cdot(-1/2)+1 = -1+1 = 0$ and $2\cdot(\pi-4)/8+1 = 2\pi/8 - 8/8 + 1 = \pi/4 - 1 + 1 = \pi/4$, *i.e.*: $\begin{matrix} x & -1/2 & (\pi-4)/8 \\ u & 0 & \pi/4 \end{matrix}$ Using the fact that $\frac{1}{\cos^2(u)} = \sec^2(u)$ it follows that:

$$\int_{-1/2}^{(\pi-4)/8} \frac{e^{\tan(2x+1)}}{\cos^2(2x+1)}\,dx = \int_0^{\pi/4} \frac{e^{\tan(u)}}{\cos^2(u)} \cdot \frac{1}{2}\,du = \frac{1}{2}\int_0^{\pi/4} e^{\tan(u)} \sec^2(u)\,du\,,$$

We will now do a second substitution, namely $w = \tan(u)$, so $\frac{dw}{du} = \frac{d}{du}\tan(u) = \sec^2(u)$, and thus $dw = \sec^2(u)\,du$. We will change the limits with this substitution as well. Note that $\cos(0) = 1$ and $\sin(0) = 0$, so $\tan(0) = \frac{\sin(0)}{\cos(0)} = \frac{0}{1} = 0$, and $\cos(\pi/4) = 1/\sqrt{2} = \sin(\pi/4)$, so $\tan(\pi/4) = \frac{\sin(\pi/4)}{\cos(\pi/4)} = \frac{1/\sqrt{2}}{1/\sqrt{2}} = 1$, *i.e.*: $\begin{matrix} u & 0 & \pi/4 \\ w & 0 & 1 \end{matrix}$ Thus:

$$\begin{aligned}\int_{-1/2}^{(\pi-4)/8} \frac{e^{\tan(2x+1)}}{\cos^2(2x+1)}\,dx &= \frac{1}{2}\int_0^{\pi/4} e^{\tan(u)} \sec^2(u)\,du = \frac{1}{2}\int_0^1 e^w\,dw \\ &= \frac{1}{2}e^w\Big|_0^1 = \frac{1}{2}e^1 - \frac{1}{2}e^0 = \frac{1}{2}e - \frac{1}{2}\cdot 1 = \frac{1}{2}(e-1) \quad \blacksquare\end{aligned}$$