Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Summer 2018
Solutions to the Actual Final Examination

Time-space: *09:00-12:00 in FPHL 117.* *Brought to you by* Стефан Біланюк.
Instructions: Do parts **A** and **B**, and, if you wish, part **C**. Show all your work and justify all your answers. *If in doubt about something,* **ask!**
Aids: Any calculator; (all sides of) one aid sheet; one (1) brain (no neuron limit).

Part A. Do all four (4) of **1**–**4**.

1. Compute $\frac{dy}{dx}$ as best you can in any *four* (4) of **a**–**f**. *[20 = 4 × 5 each]*

a. $y = 3^x$ **b.** $x^3 - y^2 = 0$ **c.** $y = x \cdot \left[\int_1^x t^2\, dt\right]$

d. $y = \dfrac{x}{x^2+2}$ **e.** $y = e^x \cos(x)$ **f.** $y = \tan^2(x)$

SOLUTIONS. **a.** We'll use the fact that $3 = e^{\ln(3)}$ and the Chain Rule:

$$\frac{dy}{dx} = \frac{d}{dx}3^x = \frac{d}{dx}\left(e^{\ln(3)}\right)^x = \frac{d}{dx}e^{\ln(3)\cdot x} = e^{\ln(3)\cdot x}\cdot\frac{d}{dx}\left(\ln(3)\cdot x\right) = 3^x \cdot \ln(3) \quad \square$$

b. *(Solve for y.* We'll solve for y and then use the Power Rule. First:

$$x^3 - y^2 = 0 \implies y^2 = x^3 \implies y = \pm x^{3/2}$$

Second:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\pm x^{3/2}\right) = \pm\frac{3}{2}x^{1/2} = \pm\frac{3}{2}\sqrt{x} \quad \square$$

b. *(Implicit differentiation.)* Here goes, using the Chain and Power Rules:

$$\frac{d}{dx}\left(x^3 - y^2\right) = \frac{d}{dx}0 = 0 \implies 0 = \frac{d}{dx}x^3 - \frac{d}{dx}y^2 = 3x^2 - \left(\frac{d}{dy}y^2\right)\cdot\frac{dy}{dx}$$
$$\implies 0 = 3x^2 - 2y\frac{dy}{dx} \implies \frac{dy}{dx} = \frac{3x^2}{2y}$$

If one were to solve for y in the original equation and plug it in to the answer above, one would get the same answer as obtained in the previous solution to **b**. $\square$

c. *(Integrate first.)* We work out y as a function of x and then differentiate it. The Power Rules for derivatives and integrals rule! First:

$$y = x \cdot \left[\int_1^x t^2\, dt\right] = x \cdot \left[\left.\frac{t^3}{3}\right|_1^x\right] = x \cdot \left[\frac{x^3}{3} - \frac{1^3}{3}\right] = x \cdot \frac{1}{3}\left[x^3 - 1\right] = \frac{1}{3}x^4 - \frac{1}{3}x$$

Second:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{3}x^4 - \frac{1}{3}x\right) = \frac{1}{3}\cdot 4x^3 - \frac{1}{3}\cdot 1 = \frac{4}{3}x^3 - \frac{1}{3} \quad \square$$

c. *(Using the Fundamental Theorem.)* We'll use the Product Rule and the Fundamental Theorem of Calculus:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x\cdot\left[\int_1^x t^2\,dt\right]\right) = \left(\frac{d}{dx}x\right)\cdot\left[\int_1^x t^2\,dt\right] + x\cdot\frac{d}{dx}\left[\int_1^x t^2\,dt\right] \\ &= 1\cdot\left[\left.\frac{t^3}{3}\right|_1^x\right] + x\cdot x^2 = \left[\frac{1}{3}x^3 - \frac{1}{3}1^2\right] + x^3 = \frac{4}{3}x^3 - \frac{1}{3}\end{aligned}$$

Note that we still had to integrate to finish the job here. $\square$

d. Our main tool here will be the Quotient Rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x}{x^2+2}\right) = \frac{\left(\frac{d}{dx}x\right)\cdot\left(x^2+2\right) - x\cdot\frac{d}{dx}\left(x^2+2\right)}{\left(x^2+2\right)^2} \\ &= \frac{1\cdot\left(x^2+2\right) - x\cdot(2x+0)}{\left(x^2+2\right)^2} = \frac{x^2+2-2x^2}{\left(x^2+2\right)^2} = \frac{2-x^2}{\left(x^2+2\right)^2} \quad \square\end{aligned}$$

e. This is a job for the Product Rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(e^x\cos(x)\right) = \left(\frac{d}{dx}e^x\right)\cdot\cos(x) + e^x\cdot\left(\frac{d}{dx}\cos(x)\right) \\ &= e^x\cos(x) + e^x\left(-\sin(x)\right) = e^x\left(\cos(x) - \sin(x)\right) \quad \square\end{aligned}$$

f. Chain Rule and Power Rule:

$$\frac{dy}{dx} = \frac{d}{dx}\tan^2(x) = \frac{d}{dx}\left(\tan(x)\right)^2 = 2\tan(x)\cdot\frac{d}{dx}\tan(x) = 2\tan(x)\sec^2(x) \quad \square$$

2. Evaluate any *four* (4) of the integrals **a**–**f**. *[20 = 4 × 5 each]*

$$\textbf{a.}\ \int x\arctan(x)\,dx \quad \textbf{b.}\ \int_0^{\pi/4}\cos(2t)\,dt \quad \textbf{c.}\ \int_e^{e^e}\frac{1}{w\ln(w)}\,dw$$

$$\textbf{d.}\ \int\frac{1}{(2y+1)^2}\,dy \quad \textbf{e.}\ \int z\tan(z)\,dz \quad \textbf{f.}\ \int_0^1 4u\,e^{u^2}\,du$$

SOLUTIONS. **a.** This is a task for integration by parts and some underhanded algebra. We will let $u=\arctan(x)$ and $v'=x$, so $u'=\frac{d}{dx}\arctan(x)=\frac{1}{x^2+1}$ and $v=\int x\,dx=\frac{x^2}{2}$.

$$\begin{aligned}\int x\arctan(x)\,dx &= uv-\int u'v\,dx = \arctan(x)\cdot\frac{x^2}{2}-\int\frac{x^2}{2}\cdot\frac{1}{x^2+1}\,dx\\ &=\frac{1}{2}x^2\arctan(x)-\frac{1}{2}\int\frac{x^2}{x^2+1}\,dx\\ &=\frac{1}{2}x^2\arctan(x)-\frac{1}{2}\int\frac{x^2+1-1}{x^2+1}\,dx\\ &=\frac{1}{2}x^2\arctan(x)-\frac{1}{2}\int\left(\frac{x^2+1}{x^2+1}-\frac{1}{x^2+1}\right)\,dx\\ &=\frac{1}{2}x^2\arctan(x)-\frac{1}{2}\int 1\,dx+\frac{1}{2}\int\frac{1}{x^2+1}\,dx\\ &=\frac{1}{2}x^2\arctan(x)-\frac{1}{2}x+\frac{1}{2}\arctan(x)+C \quad \square\end{aligned}$$

b. We will use the substitution $u=2t$, so $du=2\,dt$ and $dt=\frac{1}{2}\,du$, and also $\begin{matrix} t & 0 & \pi/4\\ u & 0 & \pi/2\end{matrix}$.

$$\begin{aligned}\int_0^{\pi/4}\cos(2t)\,dt &= \int_0^{\pi/2}\cos(u)\,\frac{1}{2}\,du=\frac{1}{2}\sin(u)\Big|_0^{\pi/2}\\ &=\frac{1}{2}\sin\left(\frac{\pi}{2}\right)-\frac{1}{2}\sin(0)=\frac{1}{2}\cdot1-\frac{1}{2}\cdot0=\frac{1}{2}\quad\square\end{aligned}$$

c. We will use the substitution $z=\ln(w)$, so $dz=\frac{1}{w}\,dw$ and $\begin{matrix} w & e & e^e\\ z & 1 & e\end{matrix}$.

$$\int_e^{e^e}\frac{1}{w\ln(w)}\,dw=\int_1^e\frac{1}{z}\,dz=\ln(z)|_1^e=\ln(e)-\ln(0)=1-0=1\quad\square$$

d. We will use the substitution $u=2y+1$, so $du=2\,dy$ and $dy=\frac{1}{2}\,du$.

$$\begin{aligned}\int\frac{1}{(2y+1)^2}\,dy &= \int\frac{1}{u^2}\,\frac{1}{2}\,du=\frac{1}{2}\int u^{-2}\,du=\frac{1}{2}\cdot\frac{u^{-1}}{-1}+C\\ &=-\frac{1}{2u}+C=-\frac{1}{2(2y+1)}+C=-\frac{1}{4y+2}+C\quad\square\end{aligned}$$

e. This is beyond ugly: $z\tan(z)$ has no antiderivative in "elementary terms", that is, as a function built out of the familiar functions in the usual ways. At this level, the most reasonable thing to try is integration by parts, in which case you either go around in circles or go crazy trying to integrate things like $\ln(\cos(z))$. One can get somewhere useful either by doing this in terms of suitable power series, or by using non-"elementary" functions that are usually defined in terms of integrals. Credit was given to reasonable attempts using the techniques developed in MATH 1110H. $\square$

f. We will use the substitution $s = u^2$, so $ds = 2u\,du$ and $\begin{array}{ccc} u & 0 & 1 \\ s & 0 & 1 \end{array}$, as well as the fact that $4 = 2 \cdot 2$.

$$\int_0^1 4u\,e^{u^2}\,du = \int_0^1 2e^s\,ds = 2e^s\Big|_0^1 = 2e^1 - 2e^0 = 2e - 2 = 2(e-1) \quad \square$$

3. Do any *four* (4) of **a**–**f**. *[20 = 4 × 5 each]*

a. Find the equation of the tangent line to $y = \sin(x)$ at $x = \dfrac{\pi}{2}$.

b. Compute $\displaystyle\lim_{x\to\infty} \frac{\ln\left(x^2\right)}{x}$.

c. Use the limit definition of the derivative to verify that $\dfrac{d}{dx}e^x = e^x$ for all x.

[You may assume that $\displaystyle\lim_{h\to 0} \frac{e^h - 1}{h} = 1$.]

d. Find the minimum value of $f(x) = xe^x$, if it has one.

e. Use the ε–δ definition of limits to verify that $\displaystyle\lim_{x\to 3}(4 - x) = 1$.

f. Sketch the region between $y = x^3$ and $y = x$ for $-1 \leq x \leq 0$ and find its area.

SOLUTIONS. **a.** The slope of the tangent line is given by

$$m = \left.\frac{dy}{dx}\right|_{x=\pi/2} = \left.\frac{d}{dx}\sin(x)\right|_{x=\pi/2} = \cos\left(\frac{\pi}{2}\right) = 0\,.$$

The equation of the tangent line is therefore $y = 0x + b = b$ for some constant b. To determine b, observe that at $\dfrac{\pi}{2}$, $b = y = \sin\left(\frac{\pi}{2}\right) = 1$. Thus the equation of the tangent line to $y = \sin(x)$ at $x = \frac{\pi}{2}$ is $y = 1$. □

b. Both $\ln\left(x^2\right) = 2\ln(x) \to \infty$ and $x \to \infty$ as $x \to \infty$, so we can apply l'Hôpital's Rule to compute the given limit:

$$\lim_{x\to\infty} \frac{\ln\left(x^2\right)}{x} = \lim_{x\to\infty} \frac{2\ln(x)}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}\ln\left(x^2\right)}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} \lim_{x\to\infty} \frac{1}{x} = 0 \quad \square$$

c. Here we go:

$$\begin{aligned}
\frac{d}{dx}e^x &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x e^h - e^x}{h} \\
&= \lim_{h\to 0} \frac{e^x\left(e^h - 1\right)}{h} = e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x \quad \square
\end{aligned}$$

d. $f(x) = xe^x$ is defined and continuous for all $x \in \mathbb{R}$, so we need to check its limit in both directions and any critical points. First:

$$\begin{aligned}
\lim_{x\to-\infty} xe^x &= \lim_{x\to-\infty} \frac{x \ \to \infty}{e^{-x} \to \infty} = \lim_{x\to-\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^{-x}} \quad \text{by l'Hôpital's Rule} \\
&= \lim_{x\to-\infty} \frac{1}{-e^{-x}} = 0 \quad (\text{since } -e^{-x} \to -\infty \text{ as } x \to -\infty) \\
\lim_{x\to+\infty} xe^x &= +\infty \quad (\text{since } x \to +\infty \text{ and } e^x \to +\infty \text{ as } x \to +\infty)
\end{aligned}$$

Second: $f'(x) = \dfrac{d}{dx}xe^x = 1 \cdot e^x + x \cdot e^x = (1+x)e^x$ with the help of the Product Rule. Since $e^x > 0$ for all x, $f'(x) = 0$ only when $1 + x = 0$, *i.e.* when $x = -1$, so there is only one critical point. Note that $f(-1) = -1 \cdot e^{-1} = -\dfrac{1}{e}$.

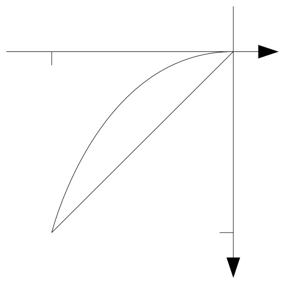
Since $f(-1) = -\dfrac{1}{e}$ is less than both 0 and ∞, it follows from the above that it is the minimum value of $f(x) = xe^x$. □

e. According to the ε–δ definition of limits, $\lim_{x \to 3}(4 - x) = 1$ means that for every $\varepsilon > 0$ there is some $\delta > 0$ so that for all x with $|x - 3| < \delta$, we have $|(4 - x) - 1| < \varepsilon$. Suppose, then, that we are given an $\varepsilon > 0$. We will reverse-engineer the corresponding δ:

$$|(4 - x) - 1| < \varepsilon \;\Leftrightarrow\; |3 - x| < \varepsilon \;\Leftrightarrow\; |x - 3| < \varepsilon$$

If we set $\delta = \varepsilon$, then, by tracing the equivalences above from right to left, we see that $|x - 3| < \delta$ implies that $|(4 - x) - 1| < \varepsilon$. Since this process works for any and all $\varepsilon > 0$, it follows that $\lim_{x \to 3}(4 - x) = 1$ by the ε–δ definition of limits. □

f. Here is a sketch of the region:



Note that $y = x^3$ is above $y = x$ when x is between -1 and 0. It follows that the area between the two curves is:

$$\begin{aligned} \text{Area} &= \int_{-1}^{0} \left(x^3 - x\right)\, dx = \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Bigg|_{-1}^{0} = \left(\frac{0^4}{4} - \frac{0^2}{2}\right) - \left(\frac{(-1)^4}{4} - \frac{(-1)^2}{2}\right) \\ &= 0 - \left(\frac{1}{4} - \frac{1}{2}\right) = -\left(-\frac{1}{4}\right) = \frac{1}{4} \quad \blacksquare \end{aligned}$$

4. Find the domain and all intercepts, vertical and horizontal asymptotes, and maximum, minimum, and inflection points of $f(x) = \dfrac{x^2+1}{x}$, and sketch its graph. *[14]*

SOLUTION. *i. (Domain)* $f(x) = \dfrac{x^2+1}{x}$ is defined whenever $x \neq 0$, so it has domain $\{ x \in \mathbb{R} \mid x \neq 0 \} = (-\infty, 0) \cup (0, \infty)$.

Note that $f(x) = \dfrac{x^2+1}{x} = x + \dfrac{1}{x}$ for all $x \neq 0$. We will use whichever form is more convenient for each part of the problem. Also, since $f(x)$ is a rational function, it is continuous and differentiable everywhere it is defined.

ii. (Intercepts) There is no y-intercept because $f(0)$ is undefined. For an x-intercept, we would have to have that $f(x) = \dfrac{x^2+1}{x} = 0$, which would require that numerator be equal to 0, *i.e.* that $x^2+1=0$. Since $x^2+1 \geq 1 > 0$ for all x, the numerator is never 0, and so there is no x-intercept either.

iii. (Vertical Asymptotes) Since $f(x)$ is defined and continuous everywhere except at $x=0$, we only need to check what happens at this point.

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(x + \frac{1}{x} \right) = 0 + \infty = +\infty$$
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(x + \frac{1}{x} \right) = 0 - \infty = -\infty$$

Thus $f(x)$ has a vertical asymptote at $x=0$, aproaching $-\infty$ when approaching $x=0$ from the left and $+\infty$ when approaching $x=0$ from the right.

iv. (Horizontal Asymptotes) We check what happens as $x \to \pm\infty$.

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \left(x + \frac{1}{x} \right) = +\infty + 0 = +\infty$$
$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \left(x + \frac{1}{x} \right) = -\infty + 0 = -\infty$$

Since $f(x)$ does not approach a real number when $x \to -\infty$ or $x \to +\infty$, $f(x)$ has no horizontal asymptote.

v. (Maxima and Minima) $f'(x) = \dfrac{d}{dx}\left(x + \dfrac{1}{x} \right) = 1 - \dfrac{1}{x^2}$, so $f'(x) = 0$ exactly when $x^2 = 1$, *i.e.* when $x=-1$ or $x=1$, and is undefined only at $x=0$, where $f(x)$ has a vertical asymptote. Since $x^2 > 0$ whenever $x \neq 0$, it follows that $f'(x) = 1 - \dfrac{1}{x^2} < 0$, and hence that $f(x)$ is decreasing, whenever $|x| < 1$ (since then $\frac{1}{x^2} > 1$), and $f'(x) = 1 - \dfrac{1}{x^2} > 0$, and hence that $f(x)$ is increasing, whenever $|x| > 1$ (since then $\frac{1}{x^2} > 1$). As usual, we summarize these facts in a table:

x	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \infty)$
$f'(x)$	$+$	0	$-$	undef	$-$	0	$+$
$f(x)$	$\uparrow$	max	$\downarrow$	undef	$\downarrow$	min	$\uparrow$

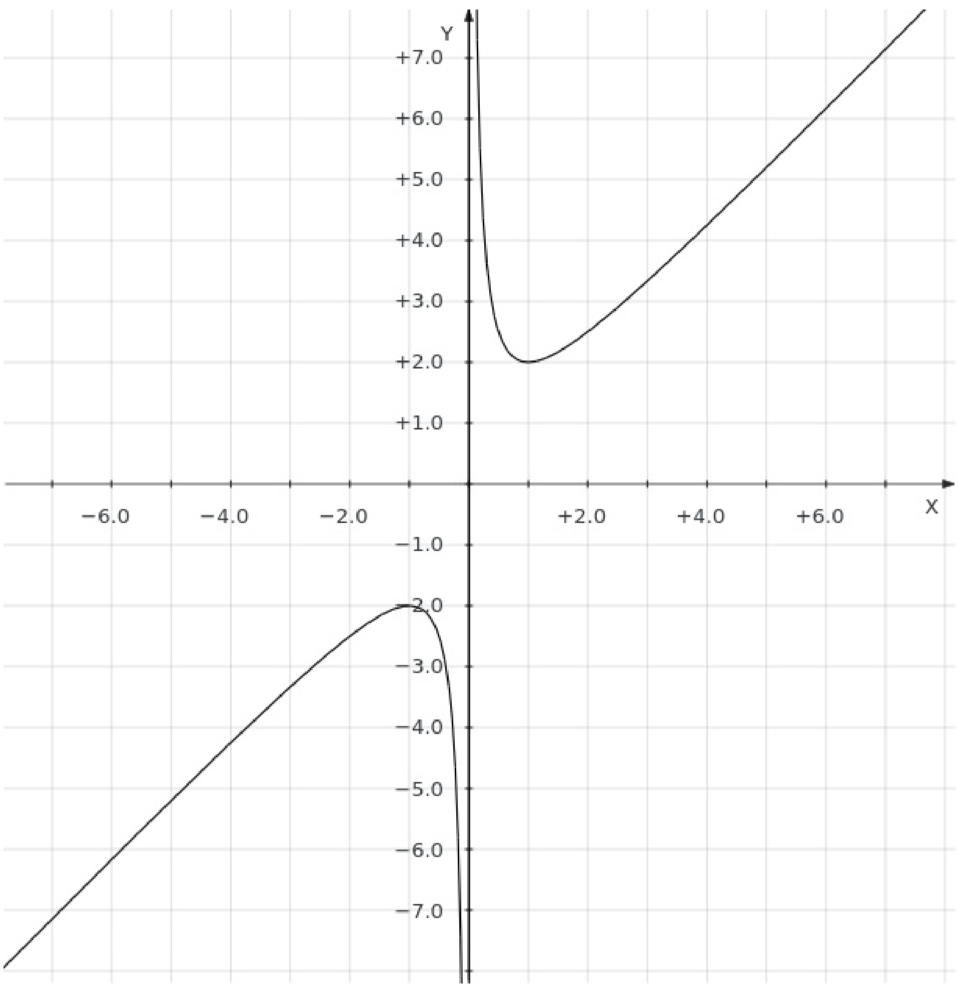
It follows that $f(x)$ has a (local) maximum of $f(-1) = -2$ at $x = -1$ and a (local) minimum of $f(1) = 2$ at $x = 1$. Since $f(x)$ reaches for both $-\infty$ and $+\infty$ by *iii* and *iv* above, nevermind that the local maximum is below the local minimum, these cannot be an absolute maximum or minimum.

vi. (Curvature and Inflection) $f''(x) = \dfrac{d^2}{dx^2}\left(x + \dfrac{1}{x}\right) = \dfrac{d}{dx}\left(1 - \dfrac{1}{x^2}\right) = -\left(-\dfrac{2}{x^3}\right) = \dfrac{2}{x^3}$, so $f''(x)$ cannot be equal to 0 when it is defined, and is undefined at $x = 0$. Since x^3 is positive or neagtive exactly when x is, $f''(x) < 0$ when $x < 0$ and $f''(x) > 0$ when $x > 0$, and thus $f(x)$ is concave down when $x < 0$ and concave up when $x > 0$. Again, we summarize all this in a table:

x	$(-\infty, 0)$	0	$(0, \infty)$
$f''(x)$	$-$	undef	$+$
$f(x)$	$\frown$	undef	$\smile$

In particular, $f(x)$ has no inflection point.

vii. (Graph) It's cheating, but here's a computer-generated plot of $y = \dfrac{x^2+1}{x} = x + \dfrac{1}{x}$:



■

Part B. Do any *two* (2) of **5–7**. *[28 = 2 × 13 each]*

5. A pebble is dropped into a still pool of water, creating a circular ripple that moves out from the point of impact at a constant rate of $2\ m/s$. How are the total length of the ripple and the area enclosed by the ripple changing after $3\ s$?

SOLUTION. Recall that the perimeter or circumference of a circle of radius r has length $2\pi r$, and that the area of the circle is πr^2. The circular ripple moving outward at a constant rate of $2\ m/s$ means that the radius is increasing at that rate, *i.e.* that $\frac{dr}{dt} = 2\ m/s$; since the pool is initially still and the ripple created when the pebble is dropped in at time $t = 0\ s$, we also have that $r(0) = 0$. It follows that

$$r(t) = \int_0^t 2\, dx = 2x\big|_0^t = 2t - 2 \cdot 0 = 2t\,.$$

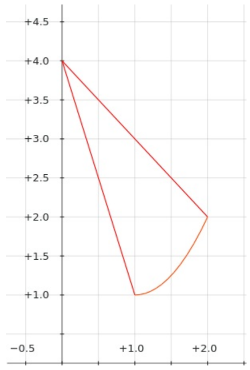
Let $P(t)$ and $A(t)$ be the length and area enclosed by the ripple at time t, as measured in seconds, respectively. Then $P(t) = 2\pi r(t) = 2\pi \cdot 2t = 4\pi t\ m$ and $A(t) = \pi r^2(t) = \pi(2t)^2 = 4\pi t^2\ m^2$, so $P'(t) = \frac{d}{dt}4\pi t = 4\pi\ m/s$ and $A'(t) = \frac{d}{dt}4\pi t^2 = 4\pi \cdot 2t = 8\pi t\ m^2/s$. After 3 seconds, *i.e.* at $t = 3\ s$, the total length of the ripple is changing at a rate of $P'(3) = 4\pi\ m/s$ and the area enclosed by the ripple is changing at a rate of $A'(3) = 8\pi \cdot 3 = 24\ m^2/s$. ■

6. Consider the region in the first quadrant (*i.e.* where both $x \geq 0$ and $y \geq 0$) below $y = 4 - x$, and above both $y = 4 - 3x$ and $y = x^2 - 2x + 2$. Find the coordinates of the three corners of this region, sketch this region, and compute the area of this region.

SOLUTION. First, the lines $y = 4 - x$ and $y = 4 - 3x$ intersect when $4 - x = 4 - 3x$, *i.e.* when $x = 0$; since $y = 4 - 0 = 4 - 3 \cdot 0 = 4$ when $x = 0$, the two lines intersect at the point $(0, 4)$. Second, the line $y = 4 - x$ and the parabola $y = x^2 - 2x + 2$ intersect when $4 - x = x^2 - 2x + 2$, that is, when $x^2 - x - 2 = (x - 2)(x + 1) = 0$, *i.e.* when $x = 2$ or $x - 1$, of which only $x = 2$ could give a point in the first quadrant. When $x = 2$, we have $y = 4 - 2 = 2^2 - 2 \cdot 2 + 2 = 2$, so the line $y = 4 - x$ and the parabola $y = x^2 - 2x + 2$ intersect at the point $(2, 2)$. Third, the line $y = 4 - 3x$ and the parabola $y = x^2 - 2x + 2$ intersect when $4 - 3x = x^2 - 2x + 2$, that is, when $x^2 + x - 2 = (x - 1)(x + 2) = 0$, *i.e.* when $x = 1$ or $x = -2$, of which only $x = 1$ could give a point in the first quadrant. When $x - 1$, we have $y = 4 - 3 \cdot 1 = 1^2 - 2 \cdot 1 + 2 = 1$, so the line $y = 4 - 3x$ and the parabola $y = x^2 - 2x + 2$ intersect at the point $(1, 1)$. Thus the corners of the region, from left to right, are $(0, 4)$, $(1, 1)$, and $(2, 2)$.

Cheating again, here is a computer-drawn plot of the region:

Note that the region can be divided up into two sub-regions: the sub-region below $y = 4 - x$ and above $y = 4 - 3x$ for $0 \leq x \leq 1$ and the sub-region below $y = 4 - x$ and above $y = x^2 - 2x + 2$ for $1 \leq x \leq 2$. Each of the sub-regions' areas is pretty easy to work

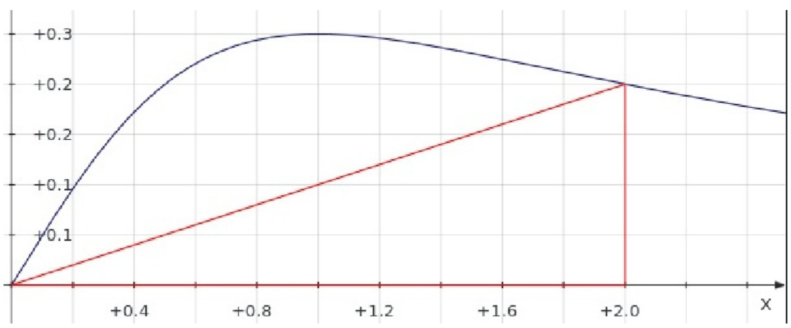


out, and the area of the region is their sum:

$$\begin{aligned}\text{Area} &= \int_0^1 [(4-x)-(4-3x)]\,dx + \int_1^2 \left[(4-x)-\left(x^2-2x+2\right)\right]\,dx \\ &= \int_0^1 2x\,dx + \int_1^2 \left(-x^2+x+2\right)\,dx = x^2\Big|_0^1 + \left(-\frac{x^3}{3}+\frac{x^2}{2}+2x\right)\Bigg|_1^2 \\ &= \left[1^2-0^2\right] + \left[\left(-\frac{2^3}{3}+\frac{2^2}{2}+2\cdot 2\right)-\left(-\frac{1^3}{3}+\frac{1^2}{2}+2\cdot 1\right)\right] \\ &= 1+\left[\frac{10}{3}-\frac{7}{6}\right] = 1+\frac{13}{6} = \frac{19}{6} \quad \blacksquare\end{aligned}$$

7. What is the maximum area of a triangle whose vertices are the points $(0,0)$, $(x,0)$, and $\left(x, \dfrac{1}{1+x^2}\right)$ for some $x \geq 0$?

SOLUTION. If $x > 0$, $(0,x)$ is directly to the right of $(0,0)$ and directly below $\left(x, \dfrac{1}{1+x^2}\right)$, so the three points form a right triangle with base $b = x-0 = x$ and height $h = \dfrac{1}{1+x^2} - 0 = \dfrac{x}{1+x^2}$. Here is what it looks like when $x = 2$:



It follows that the area of the triangle is $A(x) = \dfrac{1}{2}bh = \dfrac{1}{2}\cdot x \cdot \dfrac{1}{1+x^2} = \dfrac{x}{2\left(1+x^2\right)}$. We wish to find the maximum of this function for $0 \leq x < \infty$, *i.e.* for $x \in [0,\infty)$. As usual, we check what happens at the ends of the interval and at critical points inside it.

First, the ends of the interval:

$$A(0) = \frac{0}{2\left(1+0^2\right)} = \frac{0}{2} = 0$$

$$\lim_{x\to\infty} A(x) = \lim_{x\to\infty} \frac{x}{2\left(1+x^2\right)} \begin{array}{l}\to\infty\\ \to\infty\end{array} \quad \text{so we can apply l'Hôpital's Rule}$$

$$= \lim_{x\to\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}2\left(1+x^2\right)} = \lim_{x\to\infty} \frac{1}{2\left(0+2x\right)} = \lim_{x\to\infty} \frac{1}{4x} \begin{array}{l}\to 1\\ \to\infty\end{array} = 0$$

Second, we check what happens at critical points inside the interval. We will be using the Quotient Rule to compute $A'(x)$:

$$A'(x) = \frac{d}{dx}\left(\frac{x}{2\left(1+x^2\right)}\right) = \frac{\left(\frac{d}{dx}x\right)\cdot 2\left(1+x^2\right) - x\cdot\left(\frac{d}{dx}2\left(1+x^2\right)\right)}{\left(2\left(1+x^2\right)\right)^2}$$

$$= \frac{1\cdot 2\left(1+x^2\right) - x\cdot 2\left(0+2x\right)}{4\left(1+x^2\right)^2} = \frac{2-2x^2}{4\left(1+x^2\right)^2} = \frac{1-x^2}{2\left(1+x^2\right)^2}$$

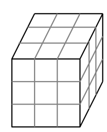
Note that $A'(x)$ is a rational function with a denominator that is always $\geq 2 > 0$, so it is defined, continuous, and differentiable for all x. $A'(x) = 0$ exactly when $1 - x^2 = 0$, *i.e.* when $x = \pm 1$. Only the critical point $x = 1$ falls in the interval $[0,\infty)$, and $A(1) = \frac{1}{2\left(1+1^2\right)} = \frac{1}{4}$.

Comparing $A(0) = 0$, $A(1) = \frac{1}{4}$, and $\lim_{x\to\infty} A(x) = 0$, we see that a triangle whose vertices are the points $(0,0)$, $(x,0)$, and $\left(x, \frac{1}{1+x^2}\right)$ for some $x \geq 0$ has maximum area $\frac{1}{4} = 0.5$, which occurs when $x = 1$. ■

[Total = 100]

Part C. Bonus problems! If you feel like it and have the time, do one or both of these.

□. A dangerously sharp tool is used to cut a cube with a side length of $3\,cm$ into 27 smaller cubes with a side length of $1\,cm$. This can be done easily with six cuts. Can it be done with fewer? (Rearranging the pieces between cuts is allowed.) If so, explain how; if not, explain why not. *[1]*



Answer. It can't be done with fewer than six cuts. You figure out why! ■

△. Write a haiku touching on calculus or mathematics in general. *[1]*

What is a haiku?

seventeen in three:
five and seven and five of
syllables in lines

Solution. Hey, I wrote the self-descriptive haiku above already … ■