Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
Trent University, Summer 2018
Solutions to the Practice Final Examination

Time: *Whatever, whenever.* *Brought to you by* Стефан Біланюк.

Instructions: Do parts **A**, **B**, and **C**, and, if you wish, part **D**. Show all your work and justify all your answers. *If in doubt about something,* **ask!**

Aids: Any calculator; (all sides of) one aid sheet; one (1) brain (no neuron limit).

Part A. Do all four (4) of **1**–**4**.

1. Compute $\frac{dy}{dx}$ as best you can in any *four* (4) of **a**–**f**. *[20 = 4 × 5 each]*

a. $y = \left(\frac{x+1}{x-1}\right)^2$ **b.** $y = \int_0^x te^{t^2}\,dt$ **c.** $\begin{array}{l} y = -\cos(t) \\ x = \sin(t) \end{array}$

d. $\ln(xy) = 0$ **e.** $y = \sin\left(\sqrt{x}\right)$ **f.** $y = x^\pi e^x$

Solutions. **a.** Power, Chain, and Quotient Rules:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x+1}{x-1}\right)^2 = 2\left(\frac{x+1}{x-1}\right)\cdot\frac{d}{dx}\left(\frac{x+1}{x-1}\right) \\ &= 2\left(\frac{x+1}{x-1}\right)\cdot\frac{\left[\frac{d}{dx}(x+1)\right](x-1)-(x+1)\left[\frac{d}{dx}(x-1)\right]}{(x-1)^2} \\ &= 2\left(\frac{x+1}{x-1}\right)\cdot\frac{1\cdot(x-1)-(x+1)\cdot 1}{(x-1)^2} = 2\left(\frac{x+1}{x-1}\right)\cdot\frac{-2}{(x-1)^2} = \frac{-4(x+1)}{(x-1)^3} \quad \blacksquare \end{aligned}$$

b. Using the Fundamental Theorem of Calculus: $\frac{dy}{dx} = \frac{d}{dx}\left(\int_0^x te^{t^2}\,dt\right) = xe^{x^2}$. ■

c. $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(-\cos(t)\right)}{\frac{d}{dt}\sin(t)} = \frac{-\left(-\sin(t)\right)}{\cos(t)} = \frac{\sin(t)}{\cos(t)} = \tan(t) = -\frac{x}{y}$. ■

d. $\ln(xy) = 0 \Rightarrow xy = 1 \Rightarrow y = \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}x^{-1} = (-1)x^{-2} = -\frac{1}{x^2}$. ■

e. Chain and Power Rules:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\sin\left(\sqrt{x}\right) = \cos\left(\sqrt{x}\right)\cdot\frac{d}{dx}\sqrt{x} = \cos\left(\sqrt{x}\right)\cdot\frac{d}{dx}x^{1/2} \\ &= \cos\left(\sqrt{x}\right)\cdot\frac{1}{2}x^{-1/2} = \frac{\cos\left(\sqrt{x}\right)}{2\sqrt{x}} \quad \blacksquare \end{aligned}$$

f. Product and Power Rules:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^\pi e^x\right) = \left[\frac{d}{dx}x^\pi\right]e^x + x^\pi\left[\frac{d}{dx}e^x\right] = \pi x^{\pi-1}e^x + x^\pi e^x = x^{\pi-1}e^x\left(\pi + x\right) \quad \blacksquare$$

2. Evaluate any *four* (4) of the integrals **a**–**f**. *[20 = 4 × 5 each]*

$$\textbf{a.}\ \int \frac{e^{\sqrt{t}}}{2\sqrt{t}}\,dt \qquad \textbf{b.}\ \int_0^{\pi/2} x\cos(x)\,dx \qquad \textbf{c.}\ \int_0^1 \arctan(y)\,dy$$

$$\textbf{d.}\ \int_0^{\ln(2)} e^{-y}\,dy \quad \textbf{e.}\ \int_0^{\sqrt{\pi}} z\cos\left(z^2\right)\,dz \quad \textbf{f.}\ \int_0^{\pi/4} \tan^2(z)\,dz$$

SOLUTIONS. **a.** We will use the substitution $u = \sqrt{t}$, so $du = \frac{1}{2\sqrt{t}}\,dt$:

$$\int \frac{e^{\sqrt{t}}}{2\sqrt{t}}\,dt = \int e^u\,du = e^u + C = e^{\sqrt{t}} + C \quad \blacksquare$$

b. We will use integration by parts with $u = x$ and $v' = \cos(x)$, so $u' = 1$ and $v = \sin(x)$:

$$\begin{aligned}\int_0^{\pi/2} x\cos(x)\,dx &= x\sin(x)\Big|_0^{\pi/2} - \int_0^{\pi/2} 1\sin(x)\,dx \\ &= \left[\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) - 0\sin(0)\right] - \left(-\cos(x)\right)\Big|_0^{\pi/2} \\ &= \left[\frac{\pi}{2}\cdot 1 - 0\cdot 0\right] + \left[\cos\left(\frac{\pi}{2}\right) - \cos(0)\right] = \frac{\pi}{2} + [0-1] = \frac{\pi}{2} - 1 \quad \blacksquare\end{aligned}$$

c. We'll use integration by parts with $u = \arctan(y)$ and $v' = 1$, so $u' = \frac{1}{1+y^2}$ and $v = y$. The remaining integral will be done using the substitution $w = 1 + y^2$, so $dw = 2y\,dy$, and thus $y\,dy = \frac{1}{2}\,dw$, and $\begin{matrix} y & 0 & 1 \\ w & 1 & 2 \end{matrix}$.

$$\begin{aligned}\int_0^1 \arctan(y)\,dy &= \int_0^1 uv'\,dy = uv\Big|_0^1 - \int_0^1 u'v\,dy = y\arctan(y)\Big|_0^1 - \int_0^1 \frac{y}{1+y^2}\,dy \\ &= [1\arctan(1) - 0\arctan(0)] - \int_1^2 \frac{1}{w}\frac{1}{2}\,dw = \left[\frac{\pi}{4} - 0\right] - \frac{1}{2}\ln\left(\frac{1}{w}\right)\Bigg|_1^2 \\ &= \frac{\pi}{4} - \left[\frac{1}{2}\ln\left(\frac{1}{2}\right) - \frac{1}{2}\ln\left(\frac{1}{1}\right)\right] = \frac{\pi}{4} - \frac{1}{2}\ln\left(\frac{1}{2}\right) \quad \blacksquare\end{aligned}$$

d. We will use the substitution $s = -y$, so $ds = -1\,dy$ and $dy = -1\,ds$, and $\begin{matrix} y & 0 & \ln(2) \\ s & 0 & -\ln(2) \end{matrix}$.

$$\begin{aligned}\int_0^{\ln(2)} e^{-y}\,dy &= \int_0^{-\ln(2)} e^s\,(-1)\,ds = -e^s\Big|_0^{-\ln(2)} \\ &= -e^{-\ln(2)} - \left(-e^0\right) = -\frac{1}{e^{\ln(2)}} - (-1) = -\frac{1}{2} + 1 = \frac{1}{2} \quad \blacksquare\end{aligned}$$

e. We'll use the substitution $w = z^2$, so $dw = 2z\,dz$ and thus $z\,dz = \frac{1}{2}\,dw$, and $\begin{matrix} z & 0 & \sqrt{\pi} \\ w & 0 & \pi \end{matrix}$.

$$\int_0^{\sqrt{\pi}} z\cos\left(z^2\right)\,dz = \int_0^{\pi} \cos(w)\cdot\frac{1}{2}\,dw = \frac{1}{2}\sin(w)\Big|_0^{\pi} = \frac{1}{2}\sin(\pi) - \frac{1}{2}\sin(0) = 0 - 0 = 0 \quad \blacksquare$$

f. We will use the trigonometric identity $\tan^2(z) = \sec^2(z) - 1$.

$$\int_0^{\pi/4} \tan^2(z)\, dz = \int_0^{\pi/4} \left[\sec^2(z) - 1\right]\, dz = \left[\tan(z) - z\right]\Big|_0^{\pi/4}$$
$$= \left[1 - \frac{\pi}{4}\right] - [0 - 0] = 1 - \frac{\pi}{4} \quad \blacksquare$$

3. Do any *four* (4) of **a–g**. *[20 = 4 × 5 each]*

- **a.** Let $f(x) = x^2 + 1$ and compute $f'(1)$ using the limit definition of the derivative.
- **b.** Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x \to 0} (2x - 1) = -1$.
- **c.** Compute $\lim_{n \to \infty} \frac{n^2}{e^n}$.
- **d.** Sketch the region between $y = x^2$ and $y = \sqrt{x}$, $0 \le x \le 1$, and find its area.
- **e.** Find the equation of the tangent line to $y = \cos(x)$ at $x = \frac{\pi}{4}$.
- **f.** Find the number b such that $\int_0^b (2x + 1)\, dx = 2$.

Solutions. **a.** Here goes:

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\left[(1+h)^2 + 1\right] - \left[1^2 + 1\right]}{h}$$
$$= \lim_{h \to 0} \frac{\left[1 + 2h + h^2 + 1\right] - 2}{h} = \lim_{h \to 0} \frac{2h + h^2}{h} = \lim_{h \to 0} (2 + h) = 2 + 0 = 2 \quad \blacksquare$$

b. We need to verify that for every $\varepsilon > 0$, there is some $\delta > 0$, such that if $|x - 0| < \delta$, then $|(2x - 1) - (-1)| < \varepsilon$. As usual, we will try to reverse-engineer the necessary δ from ε. Suppose an $\varepsilon > 0$ is given. Then

$$|(2x - 1) - (-1)| < \varepsilon \Leftrightarrow |2x - 1 + 1| < \varepsilon \Leftrightarrow |2x| < \varepsilon \Leftrightarrow |x| < \frac{\varepsilon}{2} \Leftrightarrow |x - 0| < \frac{\varepsilon}{2},$$

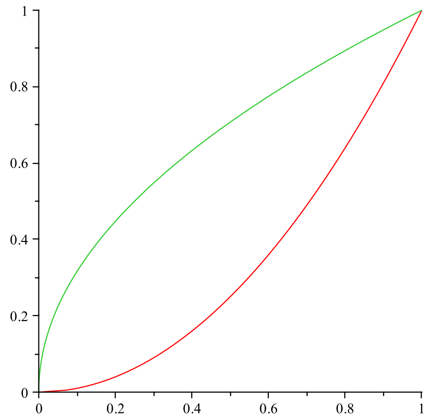
so $\delta = \frac{\varepsilon}{2}$ will do the job. Note that every step of our reverse-engineering process above is reversible, so if $|x - 0| < \delta = \frac{\varepsilon}{2}$, then $|(2x - 1) - (-1)| < \varepsilon$. $\blacksquare$

c. Here goes, using l'Hôpital's Rule twice:

$$\lim_{n \to \infty} \frac{n^2}{e^n} = \lim_{x \to \infty} \frac{x^2 \to \infty}{e^x \to \infty} = \lim_{x \to \infty} \frac{\frac{d}{dx} x^2}{\frac{d}{dx} e^x} = \lim_{x \to \infty} \frac{2x \to \infty}{e^x \to \infty}$$
$$= \lim_{x \to \infty} \frac{\frac{d}{dx} 2x}{\frac{d}{dx} e^x} = \lim_{x \to \infty} \frac{2 \to 2}{e^x \to \infty} = 0 \quad \blacksquare$$

d. Here are the curves, as plotted by `Maple`:

```
> plot([[sqrt(t),t,t=0..1],[t^2,t,t=0..1]]s)
```



The two curves intersect at $x = 0$ and $x = 1$; between these two points, $\sqrt{x} \geq x^2$. It follows that the area between the curves is given by:

$$\begin{aligned} \text{Area} &= \int_0^1 \left(\sqrt{x} - x^2\right)\, dx = \int_0^1 \left(x^{1/2} - x^2\right)\, dx = \left(\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right)\Big|_0^1 \\ &= \left(\frac{2}{3}1^{3/2} - \frac{1}{3}1^3\right) - \left(\frac{2}{3}0^{3/2} - \frac{1}{3}0^3\right) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \qquad \blacksquare \end{aligned}$$

e. $\frac{dy}{dx} = \frac{d}{dx}\cos(x) = -\sin(x)$, so the slope of the tangent line to $y = \cos(x)$ at $x = \frac{\pi}{4}$ is $m = -\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$. The line thus has an equation of the form $y = -\frac{1}{\sqrt{2}}x + b$. To determine b, we note that when $x = \frac{\pi}{4}$, we have $y = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, so $\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}} \cdot \frac{\pi}{4} + b$. It follows that $b = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{\pi}{4} = \frac{1}{\sqrt{2}}\left(1 + \frac{\pi}{4}\right)$, and thus the equation of the tangent line is $y = -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}\left(1 + \frac{\pi}{4}\right)$. $\blacksquare$

f. Observe that:

$$\int_0^b (2x+1)\, dx = \left(x^2 + x\right)\Big|_0^b = \left(b^2 + b\right) - \left(0^2 + 0\right) b^2 + b$$

We therefore need to find the number b satisfying the equation $b^2 + b = 2$, *i.e.* $b^2 + b - 2 = 0$. Using the quadratic equation, it follows that:

$$b = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-2)}}{2 \cdot 1} = \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2} = \begin{cases} \frac{2}{2} \\ -\frac{4}{2} \end{cases} = \begin{cases} 1 \\ -2 \end{cases}$$

There are thus two possible answers, $b = 1$ and $b = -2$. Note that a definite integral may still make sense even if the "lower" limit is actually greater that the "upper" limit of the integral. $\blacksquare$

4. Find the domain and any and all intercepts, vertical and horizontal asymptotes, and maximum, minimum, and inflection points of $f(x) = e^{-x^2}$, and sketch its graph.

SOLUTION. We'll run through the usual checklist and then graph $f(x) = e^{-x^2}$:

i. Domain. Note that both $g(x) = e^x$ and $h(x) = -x^2$ are defined and continuous for all x. It follows that $f(x) = g(h(x)) = e^{-x^2}$ is also defined and continuous for all x. Thus the domain of $f(x)$ is all of $\mathbb{R}$.

ii. Intercepts. Since $g(x) = e^x$ is never 0, $f(x) = e^{-x^2}$ can never equal 0 either, so it has no x-intercepts. For the y-intercept, simply note that $f(0) = e^{-0^2} = e^0 = 1$.

iii. Vertical asymptotes. $f(x) = e^{-x^2}$ is defined and continuous for all x, so it cannot have any vertical asymptotes.

iv. Horizontal asymptotes. We check for horizontal asymptotes:

$$\lim_{x\to\infty} e^{-x^2} = \lim_{x\to\infty} \frac{1}{e^{x^2}} = 0 \quad \text{and} \quad \lim_{x\to-\infty} e^{-x^2} = \lim_{x\to-\infty} \frac{1}{e^{x^2}} = 0\,,$$

since $e^{x^2} \to \infty$ as $x^2 \to \infty$, which happens as $x \to \pm\infty$. Thus $f(x) = e^{-x^2}$ has the horizontal asymptote $y = 0$ in both directions.

v. Maxima and minima. $f'(x) = e^{-x^2} \dfrac{d}{dx}\left(-x^2\right) = -2xe^{-x^2}$, which equals 0 exactly when $x = 0$ because $-2e^{-x^2} \neq 0$ for all x. Note that this is the only critical point. Since $e^{-x^2} > 0$ for all x, $f'(x) = -2xe^{-x^2} > 0$ when $x < 0$ and < 0 when $x > 0$, so $f(x) = e^{-x^2}$ is increasing for $x < 0$ and decreasing for $x > 0$. Thus $x = 0$ is an (absolute!) maximum point of $f(x)$, which has no minimum points. We summarize all this in the usual table:

x	$(-\infty, 0)$	0	$(0, \infty)$
$f'(x)$	$+$	0	1
$f(x)$	$\uparrow$	max	$\downarrow$

vi. Inflection points.

$$f''(x) = \frac{d}{dx}\left(-2xe^{-x^2}\right) = -2e^{-x^2} - 2x\frac{d}{dx}\left(-x^2\right)$$
$$= -2e^{-x^2} - 2x \cdot \left(-2xe^{-x^2}\right) = \left(4x^2 - 2\right)e^{-x^2}\,,$$

which equals 0 exactly when $4x^2 - 2 = 0$, *i.e.* when $x = \pm\frac{1}{\sqrt{2}}$, because $e^{-x^2} \neq 0$ for all x. Since $e^{-x^2} > 0$ for all x, $f''(x) = \left(4x^2 - 2\right)e^{-x^2} > 0$ exactly when $4x^2 - 2 > 0$, *i.e.* when $|x| > \frac{1}{\sqrt{2}}$, and is < 0 exactly when $4x^2 - 2 < 0$, *i.e.* when $|x| < \frac{1}{\sqrt{2}}$. Thus $f(x) = e^{-x^2}$ is concave up on $\left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$ and concave down on $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Thus $f(x) = e^{-x^2}$ has two inflection points, at $x = \pm\frac{1}{\sqrt{2}}$. We summarize all this in the usual table:

x	$\left(-\infty, -\frac{1}{\sqrt{2}}\right)$	$-\frac{1}{\sqrt{2}}$	$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\frac{1}{\sqrt{2}}$	$\left(\frac{1}{\sqrt{2}}, \infty\right)$
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\smile$	infl. pt.	$\frown$	infl. pt.	$\smile$

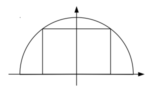
vii. Graph. Cheating just a wee bit, this graph was plotted using a program called `KAlgebra`.

■

Part B. Do any *two* (2) of **5**–**7**. *[28 = 2 × 14 each]*

5. What is the maximum area of a rectangle with its base on the x-axis and which has its two top corners on the semicircle $y = \sqrt{16 - x^2}$?

SOLUTION. A little reflection about this setup, preferably with a peek at a sketch,

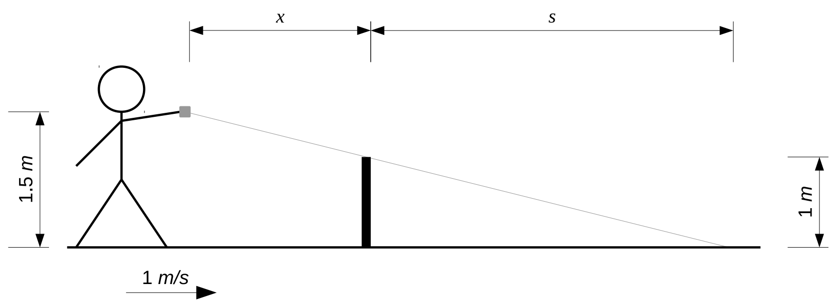


will show that the base of such a rectangle runs from $(-x, 0)$ to $(x, 0)$, while the right side runs from $(x, 0)$ to $\left(x, \sqrt{16 - x^2}\right)$. The rectangle thus has width $x - (-x) = 2x$ and height $\sqrt{16 - x^2} - 0 = \sqrt{16 - x^2}$, and thus has area $A = 2x\sqrt{16 - x^2}$. Note that areas should not be negative. Since a rectangle of width 0, which occurs when $x = 0$, has area 0, we must have $0 \leq x$, and since a rectangle of height 0, which occurs when $x = 4$, has area 0 too, we must also have $x \leq 4$. This analysis also tells us what happens at the endpoints, and that if there is a single critical point in $[0, 4]$, it must give a maximum. For critical points:

$$\begin{aligned}
\frac{dA}{dx} &= \frac{d}{dx} 2x\sqrt{16 - x^2} = \left(\frac{d}{dx} 2x\right) \cdot \sqrt{16 - x^2} + 2x \cdot \left(\frac{d}{dx}\sqrt{16 - x^2}\right) \\
&= 2\sqrt{16 - x^2} + 2x \cdot \frac{1}{2\sqrt{16 - x^2}} \cdot \left(\frac{d}{dx}\left(16 - x^2\right)\right) \\
&= 2\sqrt{16 - x^2} + \frac{x}{\sqrt{16 - x^2}} \cdot (-2x) = 2\sqrt{16 - x^2} - \frac{2x^2}{\sqrt{16 - x^2}} \\
\frac{dA}{dx} = 0 &\implies 2\sqrt{16 - x^2} - \frac{2x^2}{2\sqrt{16 - x^2}} = 0 \\
&\implies 2\sqrt{16 - x^2} \cdot \frac{\sqrt{16 - x^2}}{2} - \frac{2x^2}{2\sqrt{16 - x^2}} \cdot \frac{\sqrt{16 - x^2}}{2} = 0 \\
&\implies 16 - x^2 - 2x^2 = 16 - 3x^2 = 0 \implies x^2 = \frac{16}{3} \implies x = \pm\frac{4}{\sqrt{3}} \approx \pm 2.31
\end{aligned}$$

$x = \dfrac{4}{\sqrt{3}}$ is the only critical point in the interval $[0, 4]$, so it must give the maximum. The maximum area of a rectangle meeting the given specifications is therefore $A = 2 \cdot \dfrac{4}{\sqrt{3}} \cdot \sqrt{16 - \left(\dfrac{4}{\sqrt{3}}\right)^2} = \dfrac{8}{\sqrt{3}} \cdot \sqrt{16 - \dfrac{16}{3}} = \dfrac{8}{\sqrt{3}} \cdot \sqrt{\dfrac{32}{3}} = \dfrac{8}{\sqrt{3}} \cdot \dfrac{4\sqrt{2}}{\sqrt{3}} = \dfrac{32\sqrt{2}}{3} \approx 15.1$. Whew! ■

6. Meredith, carrying a lamp 1.5 m above the ground, walks at 1 m/s along level ground directly toward a 1 m tall post at night. How is the length of the shadow cast by the post in the lamplight changing at the instant that the lamp is 2 m from the post?



Solution. Let x be the horizontal distance between the lamp and the post, and let s be the length of the shadow, as in the slightly modified diagram above. We are given that $\frac{dx}{dt} = -1$. By the similarity of the triangles involved, $\frac{x+s}{1.5} = \frac{s}{1}$, so $x + s = 1.5s = \frac{3}{2}s$ and so $x = \frac{1}{2}s$ and $s = 2x$. It follows that $\left.\frac{ds}{dt}\right|_{x=2} = 2\left.\frac{dx}{dt}\right|_{x=2} = 2(-1) = -2\ m/s$. Thus the length of the shadow is decreasing at a rate of 2 m/s at the instant in question. Note that it changes at the same constant rate at every other instant, too. ■

7. Sand is poured onto a level floor at the rate of 60 L/min. It forms a conical pile whose height is equal to the radius of the base. How fast is the height of the pile increasing when the pile is 2 m high? [The volume of a cone of height h and base radius r is $\frac{1}{3}\pi r^2 h$.]

Solution. First, we'll use metres as out primitive unit; note that $1\ L = 0.001\ m^3$, so $60\ L/min = 0.06\ m^3/min$.

Since the height of the conical pile is always equal to the radius of the base, *i.e.* $h = r$, the volume of the cone is given by $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^2 h = \frac{\pi h^3}{3}$. It follows that

$$0.06 = \frac{dV}{dt} = \frac{d}{dt}\frac{\pi}{3}h^3 = \left(\frac{d}{dh}\frac{\pi}{3}h^3\right)\cdot\frac{dh}{dt} = \pi h^2 \cdot \frac{dh}{dt},$$

so $\frac{dh}{dt} = \frac{0.06}{\pi h^2}$ at any given instant. Plugging in $h = 2\ m$ then gives

$$\left.\frac{dh}{dt}\right|_{h=2\ m} = \frac{0.06}{\pi 2^2} = \frac{0.06}{4\pi} = \frac{0.015}{\pi}.$$

If it matters, $\frac{0.015}{\pi}\ m/min = \frac{1.5}{\pi}\ cm/min \approx 0.48\ cm/min$. ■

[Total = 100]

Part C. Bonus problems! If you feel like it and have the time, do one or both of these.

○. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$. Assuming this is so [which it is], what is the series $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = 1 + \frac{1}{9} + \frac{1}{25} + \cdots$ equal to? *[1]*

Solution. A little algebra goes a long way here. Since

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \left(1 + \frac{1}{9} + \frac{1}{25} + \cdots\right) + \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \cdots\right)$$
$$= \left(1 + \frac{1}{9} + \frac{1}{25} + \cdots\right) + \frac{1}{4}\left(1 + \frac{1}{4} + \frac{1}{9} + \cdots\right) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2},$$

it follows that $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{3}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{8}$. ■

⊙. Write a haiku touching on calculus or mathematics in general. *[1]*

What is a haiku?

seventeen in three:
five and seven and five of
syllables in lines

Solution. None given! ☺

Have some fun this summer,
and drop by next year to tell me about it!