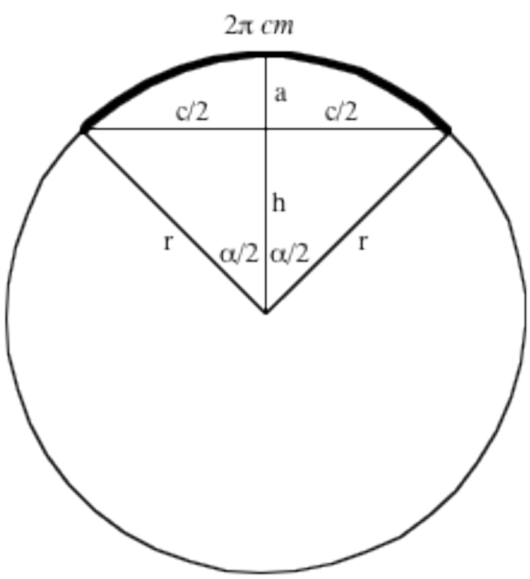
Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2012

Assignment #4
A really hairy problem!

A hair 2π *cm* long lies fully stretched out on the surface of a spherical balloon while it is being inflated. The hair does not expand or shrink during this process.

1. At the instant that the radius of the balloon is 4 *cm*, the ends of the hair are moving away from each other (in terms of straight-line distance) at a rate of 1 *cm/s*. How is the radius of the balloon changing at this instant? *[5]*

Solution. The key to understanding this set-up is that it is really only two-dimensional. If you were to cut the balloon — imagine that it wouldn't just pop! — along the hair, you would cut right through the center of the balloon. A diagram of the resulting cross-section, with various lines and angles drawn in and labelled, is given below. Note that the cross-section is just a circle with the same radius as the balloon.



We'll need some of the fundamental relationships among the various items mentioned in the diagram. First, note that, in a circle of radius r, the length of the arc subtended by an angle of α radians — that is, the length of the hair — is just $r\alpha$. (This simplicity is one of the pleasant benefits of using radians.) In our set-up, this means that

$$r\alpha = 2\pi\,.$$

This also lets us determine the angle α at the instant when $r = 4$: if $4\alpha = 2\pi$, then $\alpha = \frac{\pi}{2}$.

Second, the length c of the chord corresponding to the arc subtended by α — that is, the distance between the ends of the hair — can be computed from one the symmetric right triangles in the diagram, to get $\frac{c}{2} = r \sin\left(\frac{\alpha}{2}\right)$. Thus

$$c = 2r \sin\left(\frac{\alpha}{2}\right) ,$$

and, at the instant when $r = 4$ and $\alpha = \frac{\pi}{2}$, $c = 2 \cdot 4 \cdot \sin\left(\frac{\pi}{4}\right) = 8 \cdot \frac{1}{\sqrt{2}} = 4\sqrt{2}$.

Third, the distance a between the midpoint of the hair and the chord — that is, line between the two ends of the hair — can be obtained by subtracting the radius of the balloon from the height of the triangle whose base is the chord and whose tip is the centre of the balloon. Using one the symmetric right triangles again gives us $r^2 = h^2 + \left(\frac{c}{2}\right)^2$, so $h = \sqrt{r^2 - \frac{c^2}{4}}$. Thus

$$a = r - h = r - \sqrt{r^2 - \frac{c^2}{4}} .$$

We are told that $\frac{dc}{dt} = 1$ and need to find $\frac{dr}{dt}$. Since $r\alpha = 2\pi$, we have that $\alpha = \frac{2\pi}{r}$. So we can express c in terms of r alone,

$$c = 2r \sin\left(\frac{2\pi}{2r}\right) = 2r \sin\left(\frac{\pi}{r}\right) ,$$

and differentiate away with respect to t on both sides:

$$\begin{aligned}
\frac{dc}{dt} &= 2\frac{dr}{dt} \cdot \sin\left(\frac{\pi}{r}\right) + 2r \cdot \cos\left(\frac{\pi}{r}\right) \cdot \frac{d}{dt}\left(\frac{\pi}{r}\right) \\
&= 2\frac{dr}{dt} \cdot \sin\left(\frac{\pi}{r}\right) + 2r \cdot \cos\left(\frac{\pi}{r}\right) \cdot \frac{-\pi}{r^2} \cdot \frac{dr}{dt} \\
&= 2\frac{dr}{dt} \cdot \sin\left(\frac{\pi}{r}\right) - \frac{2\pi}{r} \cdot \cos\left(\frac{\pi}{r}\right) \cdot \frac{dr}{dt} \\
&= \frac{dr}{dt} \cdot \left[2 \sin\left(\frac{\pi}{r}\right) - \frac{2\pi}{r} \cdot \cos\left(\frac{\pi}{r}\right)\right] .
\end{aligned}$$

Since $r = 4$ at the instant in question, this gives

$$\begin{aligned}
1 &= \frac{dr}{dt} \cdot \left[2 \sin\left(\frac{\pi}{4}\right) - \frac{2\pi}{4} \cdot \cos\left(\frac{\pi}{4}\right)\right] \\
&= \frac{dr}{dt} \cdot \left[\frac{2}{\sqrt{2}} - \frac{\pi}{2} \cdot \frac{1}{\sqrt{2}}\right] \\
&= \frac{dr}{dt} \cdot \left[\frac{4 - \pi}{2\sqrt{2}}\right] .
\end{aligned}$$

It follows that the radius is changing at a rate of

$$\frac{dr}{dt} = \frac{2\sqrt{2}}{4 - \pi} .$$

That amounts to roughly 3.3. Thus, at the instant in question, the radius of the balloon is growing — note that $\frac{dr}{dt}$ is positive! — at a rate of about 3.3 cm/s. ■

2. At the same instant, how quickly is the midpoint of the hair aproaching the straight line joining the two ends? *[5]*

SOLUTION. We know $\frac{dr}{dt}$ at this instant from the solution to **1** above, and we are given $\frac{dc}{dt}$, so all we have to do is differentiate away in

$$a = r - h = r - \sqrt{r^2 - \frac{c^2}{4}}\,.$$

Then

$$\begin{aligned}
\frac{da}{dt} &= \frac{dr}{dt} - \frac{d}{dt}\left(\sqrt{r^2 - \frac{c^2}{4}}\right) \\
&= \frac{dr}{dt} - \frac{1}{2\sqrt{r^2 - \frac{c^2}{4}}} \cdot \frac{d}{dt}\left(r^2 - \frac{c^2}{4}\right) \\
&= \frac{dr}{dt} - \frac{1}{2\sqrt{r^2 - \frac{c^2}{4}}} \cdot \left(2r \cdot \frac{dr}{dt} - \frac{2c}{4} \cdot \frac{dc}{dt}\right) \\
&= \frac{dr}{dt} - \frac{2r \cdot \frac{dr}{dt} - \frac{c}{2} \cdot \frac{dc}{dt}}{2\sqrt{r^2 - \frac{c^2}{4}}}\,,
\end{aligned}$$

so, at the instant in question,

$$\begin{aligned}
\frac{da}{dt} &= \frac{2\sqrt{2}}{4-\pi} - \frac{2 \cdot 4 \cdot \frac{2\sqrt{2}}{4-\pi} - \frac{4\sqrt{2}}{2} \cdot 1}{2\sqrt{4^2 - \frac{(4\sqrt{2})^2}{4}}} \\
&= \frac{2\sqrt{2}}{4-\pi} - \frac{\frac{16\sqrt{2}}{4-\pi} - 2\sqrt{2}}{2\sqrt{16-8}} \\
&= \frac{2\sqrt{2}}{4-\pi} - \frac{2\sqrt{2} \cdot \left(\frac{8}{4-\pi} - 1\right)}{4\sqrt{2}} \\
&= \frac{2\sqrt{2}}{4-\pi} - \frac{1}{2} \cdot \left(\frac{4+\pi}{4-\pi}\right) \\
&= \frac{2\sqrt{2}}{4-\pi} - \frac{4+\pi}{2(4-\pi)} \\
&= \frac{4\sqrt{2} - 4 - \pi}{8 - 2\pi}\,.
\end{aligned}$$

That amounts to roughly -0.9. This means that the middle of the hair is getting closer — note the sign! — to the line joining the ends at a speed of about 0.9 cm/s. ■