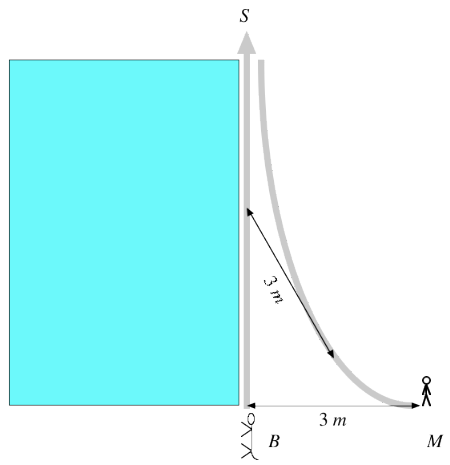
Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2012

Solution to Assignment #10
Differential Dog Day Drag

Little Max is trying to walk big dog Beau in a backyard with a rectangular pool*. With Beau keeping the $3\,m$ leash fully extended, they approach one corner of the pool along a straight line extending one side of the pool. At the instant that Beau reaches the corner, the leash is extended straight out in the direction of that side, but then Beau spots squirrel S — real name unknown! — and runs off along the other side of the pool, pulling Max after him. At any given instant, the leash is fully extended and tangent to the curve that Max is being dragged along.



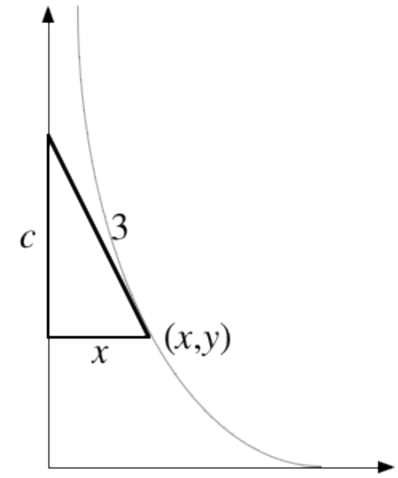
Suppose we set up a Cartesian coordinate system so that the positive y-axis is on the edge of the pool that Beau runs off along, the origin is at the corner of the pool that Beau starts running from, and Max is at $(3, 0)$ when Beau starts running.

1. Find a function $f(x)$ whose graph is the curve that Max is dragged along, with the coordinate system set up as described above. *[10]*

HINT: If Max is at (x, y) at some instant, where $y = f(x)$, the y-intercept of the tangent line always $3\,m$ from (x, y). Recall, too, that the tangent line at (x, y) has slope $m = \frac{dy}{dx} = f'(x)$. Use all this to set up an equation involving $\frac{dy}{dx}$ and then solve it for y.

SOLUTION. When Max is at $(x, f(x))$, consider the right triangle whose hypotenuse is the leash, and hence has length 3, and whose short sides are parallel to the axes. The base of this triangle, the side parallel to the x-axis, has length $x - 0 = x$; let c be the length of the other short side, the side parallel to the y-axis.

* The basic situation really happened, but no children, pets, or wildlife were harmed or forced to solve differential equations.



By the Pythagorean Theorem, $c^2 + x^2 = 3^2$, so $c = \sqrt{3^2 - x^2} = \sqrt{9 - x^2}$. The slope of the leash – which is equal to $\frac{dy}{dx}$ because the leash is tangent to the curve – is then $\frac{\text{rise}}{\text{run}} = \frac{-c}{x} = -\frac{\sqrt{9-x^2}}{x}$. (Note that the slope must be negative because the leash goes down from left to right.) It follows that $f'(x) = \frac{dy}{dx} = -\frac{\sqrt{9-x^2}}{x}$.

It remains to solve this equation for $y = f(x)$; note that we also know from the initial setup that $f(3) = 0$.

Attempt i. One way to do the job is to use `Maple`. The worksheet-style command

```
> dsolve({diff(y(x),x)=-sqrt(9-x^2)/x,y(3)=0},y(x));
```

gives the result:

$$y(x) = -\sqrt{9 - x^2} + 3\text{arctanh}\left(\frac{3}{\sqrt{9-x^2}}\right) + \frac{3}{2}I\pi$$

This solution, if you think about it, is a bit problematic: the I in the constant term represents the "imaginary" number $i = \sqrt{-1}$. You might ask yourself what it's doing here, given since we're supposed to be getting a real-valued function of the real variable x ...

Attempt ii. One can also do the job by hand using our knowledge of integration. It follows form the Fundamental Theorem of Calculus that $f'(x) = \frac{dy}{dx} = -\frac{\sqrt{9-x^2}}{x}$ and $f(3) = 0$ imply that $f(x) = \int_3^x -\frac{\sqrt{9-t^2}}{t}\, dt$. We will compute this integral using the trigonometric substitution $t = 3\sin(\theta)$, so $dt = 3\cos(\theta)\,d\theta$:

$$
\begin{aligned}
f(x) &= \int_3^x -\frac{\sqrt{9-t^2}}{t}\,dt = -\int_{t=3}^{t=x} \frac{\sqrt{9-3^2\sin^2(\theta)}}{3\sin(\theta)}\,3\cos(\theta)\,d\theta \\
&= \int_{t=x}^{t=3} \frac{3\sqrt{1-\sin^2(\theta)}}{\sin(\theta)}\cos(\theta)\,d\theta = 3\int_{t=x}^{t=3} \frac{\sqrt{\cos^2(\theta)}}{\sin(\theta)}\cos(\theta)\,d\theta \\
&= 3\int_{t=x}^{t=3} \frac{\cos(\theta)}{\sin(\theta)}\cos(\theta)\,d\theta = 3\int_{t=x}^{t=3} \frac{\cos^2(\theta)}{\sin(\theta)}\,d\theta = 3\int_{t=x}^{t=3} \frac{1-\sin^2(\theta)}{\sin(\theta)}\,d\theta \\
&= 3\int_{t=x}^{t=3} \left(\frac{1}{\sin(\theta)} - \frac{\sin^2(\theta)}{\sin(\theta)}\right)\,d\theta = 3\int_{t=x}^{t=3} \left(\csc(\theta) - \sin(\theta)\right)\,d\theta
\end{aligned}
$$

At this point we look up the antiderivative of $\csc(x)$ … ¨⌣

$$
\begin{aligned}
&= 3\left[-\ln\left(\csc(\theta)+\cot(\theta)\right) - \left(-\cos(\theta)\right)\right]\Big|_{t=x}^{t=3} \\
&= 3\left[\cos(\theta) - \ln\left(\frac{1}{\sin(\theta)} + \frac{\cos(\theta)}{\sin(\theta)}\right)\right]\Bigg|_{t=x}^{t=3}
\end{aligned}
$$

Note that $\sin(\theta) = t/3$ and $\cos(\theta) = \sqrt{1-\sin^2(\theta)} = \sqrt{1-t^2/9}$.

$$
\begin{aligned}
&= 3\left[\sqrt{1-t^2/9} - \ln\left(\frac{1}{t/3} + \frac{\sqrt{1-t^2/9}}{t/3}\right)\right]\Bigg|_{t=x}^{t=3} \\
&= \left[3\sqrt{1-t^2/9} - 3\ln\left(\frac{3}{t} + \frac{3}{t}\sqrt{1-t^2/9}\right)\right]\Bigg|_{t=x}^{t=3} \\
&= \left[\sqrt{9-t^2} - 3\ln\left(\frac{3}{t} + \frac{1}{t}\sqrt{9-t^2}\right)\right]\Bigg|_{t=x}^{t=3} \\
&= \left[\sqrt{9-t^2} - 3\ln\left(3+\sqrt{9-t^2}\right) - 3\ln\left(\frac{1}{t}\right)\right]\Bigg|_{t=x}^{t=3} \quad \text{But } \frac{1}{t} = t^{-1}, \text{ so } \ldots \\
&= \left[\sqrt{9-t^2} - 3\ln\left(3+\sqrt{9-t^2}\right) - 3(-1)\ln(t)\right]\Big|_{t=x}^{t=3} \\
&= \left[\sqrt{9-3^2} - 3\ln\left(3+\sqrt{9-3^2}\right) + 3\ln(3)\right] \\
&\quad - \left[\sqrt{9-x^2} - 3\ln\left(3+\sqrt{9-x^2}\right) + 3\ln(x)\right] \\
&= \left[0 - 3\ln(3+0) + 3\ln(3)\right] - \left[\sqrt{9-x^2} - 3\ln\left(3+\sqrt{9-x^2}\right) + 3\ln(x)\right] \\
&= 0 - \left[-3\ln\left(3+\sqrt{9-x^2}\right) + 3\ln(x) + \sqrt{9-x^2}\right] \\
&= 3\ln\left(3+\sqrt{9-x^2}\right) - 3\ln(x) - \sqrt{9-x^2}
\end{aligned}
$$

Whew! At least there are no imaginary terms … □

NOTE: The curve that occurs in this problem is called a *tractrix*.