Mathematics 1100Y – Calculus I: Calculus of one variable
Trent University, Summer 2012
Solutions to the Final Examination

Time: *14:00–17:00, on Tuesday, 7 August, 2012.* *Brought to you by* Стефан Біланюк.

Instructions: Do parts ♡, ♢, and ♣, and, if you wish, part ♠. Show all your work and justify all your answers. *If in doubt about something,* **ask!**

Aids: Calculator; up to two (≤ 2) aid sheets; at most one (≤ 1) brain.

Part ♡. Do all four (4) of **1**–**4**.

1. Compute $\dfrac{dy}{dx}$ as best you can in any *three* (3) of **a**–**f**. *[15 = 3 × 5 each]*

a. $y = \tan(2x)$ **b.** $e^x e^y = 1$ **c.** $y = e^x \cos(x)$

d. $y = \dfrac{x^2+9}{x+2}$ **e.** $\begin{array}{c} y = t+1 \\ x = \sec(t) \end{array}$ **f.** $y = \displaystyle\int_1^x e^{z+1}\,dz$

Solutions.

a. [Chain Rule] $\dfrac{dy}{dx} = \dfrac{d}{dx}\tan(2x) = \sec^2(2x)\cdot\dfrac{d}{dx}(2x) = 2\sec^2(2x)$ ■

b. [Solve for y] $e^x e^y = 1 \Longrightarrow e^y = \dfrac{1}{e^x} = e^{-x} \Longrightarrow y = -x \Longrightarrow \dfrac{dy}{dx} = -1$ ■

c. [Product Rule]

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(e^x\cos(x)\right) = \left[\frac{d}{dx}e^x\right]\cdot\cos(x) + e^x\cdot\left[\frac{d}{dx}\cos(x)\right] \\ &= e^x\cos(x) + e^x\left[-\sin(x)\right] = e^x\left(\cos(x)-\sin(x)\right) \quad ■\end{aligned}$$

d. [Quotient Rule]

$$\begin{aligned}\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+9}{x+2}\right) &= \frac{\frac{d}{dx}\left(x^2+9\right)\cdot(x+2) - \left(x^2+9\right)\cdot\frac{d}{dx}(x+2)}{\left(x^2+9\right)^2} \\ &= \frac{2x\cdot(x+2)-\left(x^2+9\right)\cdot 1}{\left(x^2+9\right)^2} = \frac{2x^2+4x-\left(x^2+9\right)\cdot 1}{\left(x^2+9\right)^2} = \frac{x^2+4x-9}{\left(x^2+9\right)^2} \quad ■\end{aligned}$$

e. [Parametrick!] $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\frac{d}{dt}(t+1)}{\frac{d}{dt}\sec(t)} = \dfrac{1}{\sec(t)\tan(t)} = \dfrac{1}{x\sqrt{1+x^2}}$ ■

f. [Fundamental Theorem of Calculus] $\dfrac{dy}{dx} = \dfrac{d}{dx}\displaystyle\int_1^x e^{z+1}\,dz = e^{x+1}$ ■

2. Evaluate any *three* (3) of the integrals **a**–**f**. *[15 = 3 × 5 each]*

a. $\displaystyle\int \frac{1}{x^3+4x}\,dx$ **b.** $\displaystyle\int_e^\infty \frac{1}{x\ln(x)}\,dx$ **c.** $\displaystyle\int \cos(2t+1)\,dt$

d. $\displaystyle\int_0^{\pi/2} \sin^2(z)\cos^3(z)\,dz$ **e.** $\displaystyle\int e^x\sec\left(e^x\right)\,dx$ **f.** $\displaystyle\int_0^1 \arctan(x)\,dx$

SOLUTIONS.

a. [Partial Fractions] $x^3+4x = x\left(x^2+4\right)$ so $\dfrac{1}{x^3+4x} = \dfrac{A}{x}+\dfrac{Bx+C}{x^2+4}$, where $1 = A\left(x^2+4\right)+(Bx+C)x = (A+B)x^2+Cx+4A$. Thus $A+B=0$, $C=0$, and $4A=1$, so $A=\frac{1}{4}$, $B=-\frac{1}{4}$, and $C=0$. Hence

$$\begin{aligned}\int \frac{1}{x^3+4x}\,dx &= \int\left(\frac{\frac{1}{4}}{x}+\frac{-\frac{1}{4}x}{x^2+4}\right)dx = \frac{1}{4}\int\frac{1}{x}\,dx - \frac{1}{4}\int\frac{x}{x^2+4}\,dx\\ &= \frac{1}{4}\ln(x)-\frac{1}{4}\int\frac{1}{u}\frac{1}{2}\,du \qquad \begin{array}{c}\text{(substituting } u=x^2+4\text{, so}\\ du=2x\,dx \text{ and } x\,dx=\frac{1}{2}\,du)\end{array}\\ &= \frac{1}{4}\ln(x)-\frac{1}{8}\ln(u)+K = \frac{1}{4}\ln(x)-\frac{1}{8}\ln\left(x^2+4\right)+K \qquad \blacksquare\end{aligned}$$

b. [Improper Integral]

$$\begin{aligned}\int_e^\infty \frac{1}{x\ln(x)}\,dx &= \lim_{t\to\infty}\int_e^t \frac{1}{x\ln(x)}\,dx = \lim_{t\to\infty}\int_1^{\ln(t)}\frac{1}{u}\,du\\ &\qquad \left(\text{substituting } u=\ln(x)\text{, so } du=\frac{1}{x}\,dx \text{ and } \begin{array}{ccc} x & e & t\\ u & 1 & \ln(t)\end{array}\right)\\ &= \lim_{t\to\infty}\ln(u)\Big|_1^{\ln(t)} = \lim_{t\to\infty}\left[\ln(\ln(t))-\ln(1)\right] = \lim_{t\to\infty}\ln(\ln(t)) = \infty\end{aligned}$$

since as $t\to\infty$, $\ln(t)\to\infty$, and hence $\ln(\ln(t))\to\infty$ too. $\blacksquare$

c. [Substitution] Let $u=2t+1$, so $du=2\,dt$ and $dt=\frac{1}{2}\,du$.

$$\int\cos(2t+1)\,dt = \int\cos(u)\,\frac{1}{2}\,du = \frac{1}{2}\sin(u)+C = \frac{1}{2}\sin(2t+1)+C \qquad \blacksquare$$

d. [Trig Identity and Substitution] Let $u = sin(z)$, so $du = \cos(z)\,dz$ and $\begin{matrix} z & 0 & \pi/2 \\ u & 0 & 1 \end{matrix}$.

$$\begin{aligned}
\int_0^{\pi/2} \sin^2(z)\cos^3(z)\,dz &= \int_0^{\pi/2} \sin^2(z)\cos^2(z)\cos(z)\,dz \\
&= \int_0^{\pi/2} \sin^2(z)\left(1-\sin^2(z)\right)\cos(z)\,dz \\
&= \int_0^1 u^2\left(1-u^2\right)\,du = \int_0^1 \left(u^2-u^4\right)\,du \\
&= \left(\frac{u^3}{3}-\frac{u^5}{5}\right)\Bigg|_0^1 = \left(\frac{1^3}{3}-\frac{1^5}{5}\right)-\left(\frac{0^3}{3}-\frac{0^5}{5}\right) \\
&= \frac{1}{3}-\frac{1}{5} = \frac{2}{15} \qquad \blacksquare
\end{aligned}$$

e. [Substitution] Let $w = e^x$, so $dw = e^x\,dx$.

$$\begin{aligned}
\int e^x \sec\left(e^x\right)\,dx &= \int \sec(w)\,dw = \ln\left(\sec(w)+\tan(w)\right)+C \\
&= \ln\left(\sec\left(e^x\right)+\tan\left(e^x\right)\right)+C \qquad \blacksquare
\end{aligned}$$

f. [Integration by Parts] Let $u = \arctan(x)$ and $v' = 1$, so $u' = \dfrac{x}{1+x^2}$ and $v = x$.

$$\int_0^1 \arctan(x)\,dx = \int_0^1 uv'\,dx = uv|_0^1 - \int_0^1 u'v\,dx = x\arctan(x)|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= 1\arctan(1) - 0\arctan(0) - \int_1^2 \frac{1}{w}\frac{1}{2}\,dw$$

(Where $w = 1+x^2$, so $dw = 2\,dx$ and $dx = \frac{1}{2}\,dw$, while $\begin{matrix} x & 0 & 1 \\ w & 1 & 2 \end{matrix}$.)

$$\begin{aligned}
&= \frac{\pi}{4} - 0 - \frac{1}{2}\ln(w)\Big|_1^2 = \frac{\pi}{4} - \left[\frac{1}{2}\ln(2)-\frac{1}{2}\ln(1)\right] = \frac{\pi}{4}-\frac{1}{2}\ln(2)+\frac{1}{2}\cdot 0 \\
&= \frac{\pi}{4}-\frac{1}{2}\ln(2) = \frac{\pi}{4}-\ln\left(\sqrt{2}\right) \qquad \blacksquare
\end{aligned}$$

3. Do any *three* (3) of **a**–**f**. *[15 = 3 × 5 each]*

a. Use the Right-hand Rule to compute the definite integral $\displaystyle\int_0^2 (x+1)\,dx$.

b. Compute $\displaystyle\lim_{n\to\infty} n\sin(n\pi)$.

c. Sketch the region between $r = 0$ and $r = \sec(\theta)$, for $0 \le \theta \le \pi/4$, in polar coordinates and find its area.

d. Find the area of the surface obtained by revolving the curve $y = x$, for $0 \le x \le 1$, about the y-axis.

e. Use the limit definition of the derivative to compute $f'(2)$ if $f(x) = x^2 + 1$.

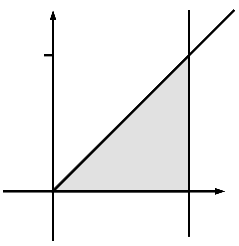
f. Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{n}{e^{2n}}$ converges or diverges.

SOLUTIONS.

a. We plug into the formula, $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{b-a}{n}\cdot f\left(a+\frac{b-a}{n}i\right)$, for the Right-Hand Rule and grind away.

$$\begin{aligned}
\int_0^2 (x+1) &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2-0}{n}\left(\left[0+\frac{2-0}{n}i\right]+1\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(\frac{2}{n}i+1\right) \\
&= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left(\frac{2}{n}i+1\right) = \lim_{n\to\infty}\frac{2}{n}\left(\left[\sum_{i=1}^{n}\frac{2}{n}i\right]+\left[\sum_{i=1}^{n}1\right]\right) \\
&= \lim_{n\to\infty}\frac{2}{n}\left(\left[\frac{2}{n}\sum_{i=1}^{n}i\right]+[n]\right) = \lim_{n\to\infty}\frac{2}{n}\left(\left[\frac{2}{n}\cdot\frac{n(n+1)}{2}\right]+n\right) \\
&= \lim_{n\to\infty}\frac{2}{n}\left([n+1]+n\right) = \lim_{n\to\infty}\frac{2}{n}(2n+1) = \lim_{n\to\infty}\left(\frac{4n}{n}+\frac{2}{n}\right) \\
&= \lim_{n\to\infty}\left(4+\frac{2}{n}\right) = 4+0 = 4 \qquad \blacksquare
\end{aligned}$$

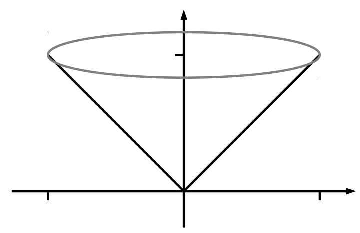
b. Since $\sin(n\pi) = 0$ for every integer n, we have $\displaystyle\lim_{n\to\infty} n\sin(n\pi) = \lim_{n\to\infty} n\cdot 0 = \lim_{n\to\infty} 0 = 0$. [Yes, a trick question!] $\blacksquare$



c. Since $x = r\cos(\theta) = \sec(\theta)\cos(\theta) = \dfrac{1}{\cos(\theta)}\cos(\theta) = 1$ for all points on the curve, it is just the vertical line $x = 1$ in Cartesian coordinates. Also, $\theta = 0$ is the positive x-axis and $\theta = \pi/4$ is the part of $y = x$ in the first quadrant, so the region looks like:

To find its area, one could certainly use the area formula for polar coordinates and evaluate the integral $\displaystyle\int_0^{\pi/4}\frac{1}{2}\sec^2(\theta)\,d\theta$, but it's easier to use the area formula for a triangle: $\frac{1}{2}bh = \frac{1}{2}\cdot 2\cdot 2 = 2\ \ldots$ $\blacksquare$

d. Here is a crude sketch of the surface, a right-circular cone with radius 1 and height 1: Since we are revolving about the y-axis, the point on the curve $y = x$ at x is revolved in a circle with radius $r = x - 0 = x = y$. We plug this and the given limits into the surface area formula and chug away, using y as the independent variable. [Because?] Note that since $y = x$ for curve, $0 \leq y \leq 1$ for the piece of the curve in question, and that $\frac{dx}{dy} = \frac{dy}{dy} = 1$.



$$\begin{aligned} SA &= \int_a^b 2\pi r \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 2\pi y \sqrt{1 + 1^2}\, dy = 2\sqrt{2}\pi \int_0^1 y\, dy \\ &= 2\sqrt{2}\pi \frac{y^2}{2}\Big|_0^1 = 2\sqrt{2}\pi \frac{1^2}{2} - 2\sqrt{2}\pi \frac{0^2}{2} = \sqrt{2}\pi \qquad \blacksquare \end{aligned}$$

e. Recall that $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, so:

$$\begin{aligned} f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{\left[(2+h)^2 + 1\right] - \left[2^2 + 1\right]}{h} \\ &= \lim_{h \to 0} \frac{\left[4 + 4h + h^2 + 1\right] - [4 + 1]}{h} = \lim_{h \to 0} \frac{4h + h^2}{h} \\ &= \lim_{h \to 0} (4 + h) = 4 + 0 = 4 \qquad \blacksquare \end{aligned}$$

f. We will use the Ratio Test to determine whether $\sum_{n=0}^{\infty} \frac{n}{e^{2n}}$ converges or diverges.

$$\begin{aligned} \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \to \infty} \left| \frac{\frac{n+1}{e^{2(n+1)}}}{\frac{n}{e^{2n}}} \right| = \lim_{n \to \infty} \frac{n+1}{e^{2n+2}} \cdot \frac{e^{2n}}{n} = \lim_{n \to \infty} \frac{e^{2n}}{e^{2n+2}} \cdot \frac{n+1}{n} \\ &= \lim_{n \to \infty} \frac{1}{e^2}\left(1 + \frac{1}{n}\right) = \frac{1}{e^2}(1 + 0) = \frac{1}{e^2} < 1 \end{aligned}$$

Since $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, it follows by the Ratio Test that the given series converges. (Absolutely, too.) $\blacksquare$

4. Consider the curve $y = \dfrac{x^2}{2}$ $0 \le x \le 2$.

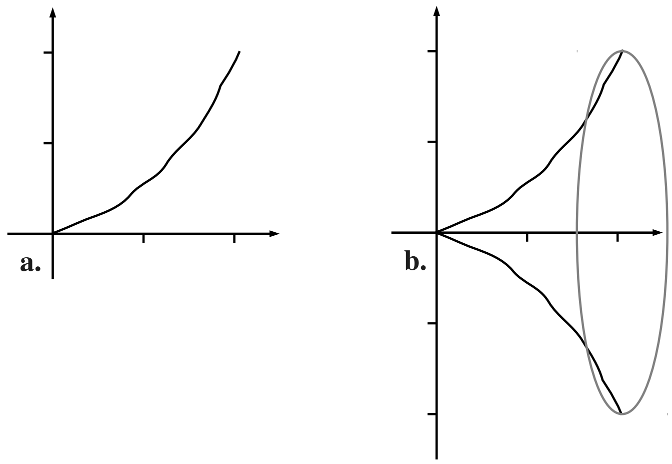
a. Sketch this curve. *[1]*

b. Sketch the surface obtained by revolving this curve about the x-axis. *[1]*

c. Compute either *i.* the length of the curve or *ii.* the area of this surface. (Not both!) *[8]*

SOLUTIONS.

a & **b.** A little shaky drawing the curve ... :-)



■

c. *i.* We plug the given limits for x and $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x^2}{2}\right) = \dfrac{2x}{2} = x$ into the formula for arc-length:

$$\text{arc-length} = \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^2 \sqrt{1+x^2}\, dx$$

Substitute $x = \tan(\theta)$, so $dx = \sec^2(\theta)\, d\theta$ and $\sqrt{1+x^2} = \sqrt{1+\tan^2(\theta)} = \sec(\theta)$.

$$= \int_{x=0}^{x=2} \sec(\theta)\, \sec^2(\theta)\, d\theta = \int_{x=0}^{x=2} \sec^3(\theta)\, d\theta = \cdots$$

See the solution to **c** *ii.* for the relevant formula.

$$= \frac{1}{2}\tan(\theta)\sec(\theta) + \frac{1}{2}\ln\left(\tan(\theta) + \sec(\theta)\right)\Big|_{x=0}^{x=2}$$

$$= \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln\left(x + \sqrt{1+x^2}\right)\Big|_0^2$$

$$= \left[\frac{1}{2}2\sqrt{1+2^2} + \frac{1}{2}\ln\left(2+\sqrt{1+2^2}\right)\right] - \left[\frac{1}{2}0\sqrt{1+0^2} + \frac{1}{2}\ln\left(0+\sqrt{1+0^2}\right)\right]$$

$$= \left[\sqrt{5} + \frac{1}{2}\ln\left(2+\sqrt{5}\right)\right] - \left[0 + \frac{1}{2}\ln(1)\right] = \sqrt{5} + \frac{1}{2}\ln\left(2+\sqrt{5}\right)$$

$$= \sqrt{5} + \ln\left(\sqrt{2+\sqrt{5}}\right)$$

(Recall that $\ln(1) = 0$ and $a\ln(b) = \ln\left(b^a\right)$). Not a nice-looking answer, is it? ■

c. *ii.* We plug the given limits for x, $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x^2}{2}\right) = \dfrac{2x}{2} = x$, and $r = y - 0 = y = \dfrac{x^2}{2}$ into the formula for surface area. Along the way we will use the reduction formula $\displaystyle\int \sec^n(\theta)\,d\theta = \frac{1}{n-1}\tan(\theta)\sec^{n-2}(\theta) + \frac{n-2}{n-1}\int \sec^{n-2}(\theta)\,d\theta$.

$$SA = \int_0^2 2\pi r\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^2 2\pi\frac{x^2}{2}\sqrt{1+x^2}\,dx = \pi\int_0^2 x^2\sqrt{1+x^2}\,dx$$

Using the same substitution as in the solution to bf c. *ii.* above gives:

$$= \pi\int_{x=0}^{x=2} \sec^2(\theta)\sec(\theta)\ \sec^2(\theta)\,d\theta = \pi\int_{x=0}^{x=2} \sec^5(\theta)\,d\theta$$

Applying the reduction formula for $\displaystyle\int \sec^n(\theta)\,d\theta$ with $n = 5$ gives:

$$= \pi\left[\frac{1}{5-1}\tan(\theta)\sec^{5-2}(\theta)\Big|_{x=0}^{x=2} + \frac{5-2}{5-1}\int_{x=0}^{x=2}\sec^{5-2}(\theta)\,d\theta\right]$$

$$= \pi\left[\frac{1}{4}\tan(\theta)\sec^3(\theta)\Big|_{x=0}^{x=2} + \frac{3}{4}\int_{x=0}^{x=2}\sec^3(\theta)\,d\theta\right]$$

Applying the reduction formula again with $n = 3$ now gives:

$$= \pi\left[\frac{1}{4}\tan(\theta)\sec^3(\theta)\Big|_{x=0}^{x=2}\right.$$

$$\left.+\frac{3}{4}\left(\frac{1}{3-1}\tan(\theta)\sec^{3-2}(\theta)\Big|_{x=0}^{x=2} + \frac{3-2}{3-1}\int_{x=0}^{x=2}\sec^{3-2}(\theta)\,d\theta\right)\right]$$

$$= \pi\left[\frac{1}{4}\tan(\theta)\sec^3(\theta)\Big|_{x=0}^{x=2} + \frac{3}{4}\left(\frac{1}{2}\tan(\theta)\sec(\theta)\Big|_{x=0}^{x=2} + \frac{1}{2}\int_{x=0}^{x=2}\sec(\theta)\,d\theta\right)\right]$$

$$= \pi\left[\frac{1}{4}\tan(\theta)\sec^3(\theta)\Big|_{x=0}^{x=2} + \frac{3}{8}\tan(\theta)\sec(\theta)\Big|_{x=0}^{x=2} + \frac{3}{8}\ln\left(\tan(\theta)+\sec(\theta)\right)\Big|_{x=0}^{x=2}\right]$$

Substituting back as in the solution to **c** *i.* gives:

$$= \pi\left[\frac{1}{4}x\left(\sqrt{1+x^2}\right)^3 + \frac{3}{8}x\sqrt{1+x^2} + \frac{3}{8}\ln\left(x+\sqrt{1+x^2}\right)\right]\Big|_0^2$$

$$= \pi\left[\frac{1}{4}2\left(\sqrt{1+2^2}\right)^3 + \frac{3}{8}2\sqrt{1+2^2} + \frac{3}{8}\ln\left(2+\sqrt{1+2^2}\right)\right]$$

$$- \pi\left[\frac{1}{4}0\left(\sqrt{1+0^2}\right)^3 + \frac{3}{8}0\sqrt{1+0^2} + \frac{3}{8}\ln\left(0+\sqrt{1+0^2}\right)\right]$$

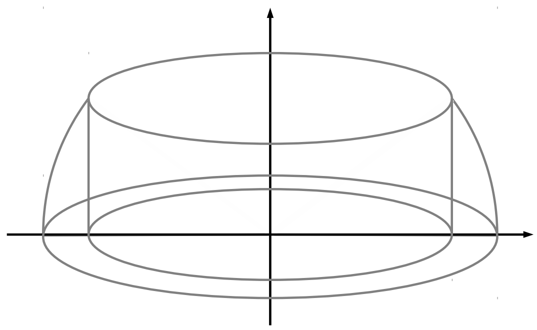
$$= \pi\left[\frac{5}{2}\sqrt{5} + \frac{3}{4}\sqrt{5} + \frac{3}{8}\ln\left(2+\sqrt{5}\right)\right]$$

Ugh!! ■

Part $\diamondsuit$. Do any *two* (2) of **5–7**. *[30 = 2 × 15 each]*

5. Sketch the solid obtained by revolving the region below $y = \sqrt{25 - x^2}$ and above $y = 0$, for $4 \le x \le 5$, about the y-axis and find its volume. *[15]*

SOLUTION. Here's a sketch of the solid:



We'll use the disk/washer method to find the volume of the solid. Since the axis of revolution was a vertical line, we will use y as the independent variable. Note that for the region in question, $0 \le y \le \sqrt{5^2 - 4^2} = 3$. The outside radius of the washer at y is $R = x = \sqrt{5^2 - y^2} = \sqrt{25 - y^2}$ and its inner radius is $r = 4$. Plugging all sthis into the appropriate volume formula gives:

$$
\begin{aligned}
V &= \int_0^3 \pi \left(R^2 - r^2\right)\, dy = \int_0^3 \pi \left(\left[\sqrt{25 - y^2}\right]^2 - 4^2\right)\, dy \\
&= \pi \int_0^3 \left(25 - y^2 - 16\right)\, dy = \pi \int_0^3 \left(9 - y^2\right)\, dy = \pi \left(9y - \frac{y^3}{3}\right)\Bigg|_0^1 \\
&= \pi \left(9 \cdot 3 - \frac{3^3}{3}\right) - \pi \left(9 \cdot 0 - \frac{0^3}{3}\right) = \pi\,(27 - 9) - \pi \cdot 0 = 18\pi \qquad \blacksquare
\end{aligned}
$$

6. Find the domain, all the intercepts, maximum, minimum, and inflection points, and all the vertical and horizontal asymptotes of $f(x) = xe^x$, and sketch its graph. *[15]*

SOLUTION. We run through the usual checklist:

i. *(Domain)* $f(x) = xe^x$ is defined and continuous for all x, since it is the product of two functions that are. $\square$

ii. *(Intercepts)* $f(0) = 0e^0 = 0$, so the y-intercept is at $y = 0$, which means it's also an x-intercept. Since $e^x > 0$ for all x, $f(x) = xe^x = 0$ only when $x = 0$, so there are no other x-intercepts. $\square$

iii. *(Vertical asymptotes)* Since $f(x)$ is defined and continuous for all x, it has no vertical asymptotes. $\square$

iv. *(Horizontal asymptotes)*

$$\lim_{x\to+\infty} xe^x = +\infty \qquad \text{since } x \to +\infty \text{ and } e^x \to +\infty \text{ as } x \to +\infty.$$

$$\lim_{x\to-\infty} xe^x \qquad \text{is harder since } x \to -\infty \text{ and } e^x \to 0 \text{ as } x \to -\infty \ldots$$

$$= \lim_{x\to-\infty} \frac{x}{e^{-x}} \quad \begin{matrix} \to -\infty \\ \to +\infty \end{matrix} \quad \text{as } x \to -\infty\text{, so we can use l'Hôpital's Rule:}$$

$$= \lim_{x\to-\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^{-x}} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} \quad \begin{matrix} \to 1 \\ \to -\infty \end{matrix} \quad = 0$$

Thus $f(x) = xe^x$ has a horizontal asymptote of $y = 0$ in the $-$'ve direction only. □

v. *(Maxima, minima, etc.)* First, using the Product Rule, $f'(x) = \frac{d}{dx}(xe^x) = 1e^x + xe^x = (1+x)e^x$. Since $e^x > 0$ for all x, it follows that $f'(x)$ is less than, equal to, or greater than 0 depending on whether $1 + x$ is, respectively, less than, equal to, or greater than 0. We build the usual table:

x	$(-\infty, -1)$	-1	$(-1, +\infty)$
$f'(x)$	$-$	0	$+$
$f(x)$	$\downarrow$	min	$\uparrow$

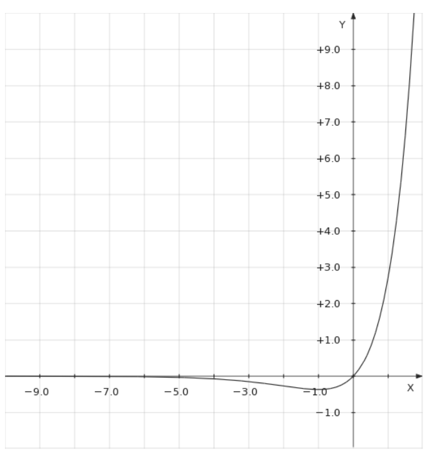
Thus the only critical point, at $x = -1$, is a minimum. Since $f(x)$ is defined, continuous, and differentiable everywhere, it is an absolute, not just a local, minimum. Note that $f(-1) = (-1)e^{-1} = -\frac{1}{e}$. □

vi. *(Curvature and inflection points)* First, using the Product Rule again, $f''(x) = \frac{d}{dx}[(1+x)e^x] = 1e^x + (1+x)e^x = (2+x)e^x$. Since $e^x > 0$ for all x, it follows that $f'(x)$ is less than, equal to, or greater than 0 depending on whether $2 + x$ is, respectively, less than, equal to, or greater than 0. We build the usual table:

x	$(-\infty, -2)$	-2	$(-2, +\infty)$
$f''(x)$	$-$	0	$+$
$f(x)$	$\frown$	infl. pt.	$\smile$

Thus $x = -2$, the only point where $f''(x) = 0$, is indeed an inflection point. □

vii. *(Graph)* Cheating slightly, here is a graph of $f(x) = xe^x$ drawn by a graphing program called `kmplot`:



$\blacksquare$

7. Freyja and Hretha sprint 100 m in lanes that are 5 m apart. The two start simultaneously at $t = 0$ s. Freyja runs at 9.6 m/s and Hretha at 10 m/s.

a. How far ahead is Hretha when she crosses the finish line? When does Freyja cross the finish line? *[1]*

b. Determine how quickly Hretha is pulling ahead as she crosses the finish line. *[1]*

c. Determine how the distance [along a direct line] between the two is changing at the instant that Hretha crosses the finish line. *[8]*

d. The two runners' starting positions and their positions at any instant thereafter form a trapezoid. How is the area of this trapezoid changing at the instant that Hretha crosses the finish line? *[5]*

SOLUTION TO **a**. If Hretha runs at 10 m/s she will cover 100 m in 10 s, in which time Freyja will have run 9.6 $m/s \times 10$ $s = 96$ m. Thus Hretha is $100 - 96 = 4$ m ahead of Freyja at the instant that she crosses the finish line.

Freyja crosses the finish line in (100 m) / (9.6 m/s) ≈ 10.417 s. ■

SOLUTION TO **b**. At any given instant after the start, including the instant Hretha crosses the finish line, she is running $10 - 9.6 = 0.4$ m/s faster than Freyja. ■

SOLUTION TO **c**. Let x denote how far ahead Hretha is and s denote the straight-line distance between the two runners at time $t > 0$. From the answer to **a** above, $x(10) = 4$; from the answer to **b** above, $\frac{dx}{dt} = 0.4$ at each instant $t > 0$. Since the lanes in which they run are 5 m apart, we also know the straight-line distance between the two runners is given by $s = \sqrt{5^2 + x^2} = \sqrt{25 - x^2}$ at each instant $t > 0$. Thus

$$\frac{ds}{dt} = \frac{d}{dt}\sqrt{25+x^2} = \frac{1}{2\sqrt{25+x^2}} \cdot \frac{d}{dt}\left(25+x^2\right) = \frac{1}{2\sqrt{25+x^2}} \cdot 2x \cdot \frac{dx}{dt} = \frac{0.4x}{\sqrt{25+x^2}}\,;$$

when Hretha crosses the finish line we have $x = 4$, so the distance between the two runners is increasing at a rate of

$$\left.\frac{ds}{dt}\right|_{x=4} = \left.\frac{0.4x}{25+x^2}\right|_{x=4} = \frac{0.4 \cdot 4}{\sqrt{25+4^2}} = \frac{1.6}{\sqrt{41}} \approx 0.249878\, m/s\,. \qquad ■$$

SOLUTION TO **d**. At any given instant, he trapezoid in question can be thought of as a rectangle plus a triangle, where the rectangle's corners are the starting points of the two runners, Freyja's position in her lane, and the position directly across from her in Hretha's lane, and the triangle's corners are Freyja's position in her lane, the position directly across from her in Hretha's lane, and Hretha's position in her own lane. The bases of the rectangle and of the triangle are a constant 5 m, the height of the triangle at any given instant is given by the distance Hretha is ahead of Freyja – *i.e.* the quantity x in the solution to **c** above, and the height of the rectangle at any given instant is given by how far from the starting line Hretha is – call this quantity y. (Note that it follows from the reasoning in the solution to **b** that $\frac{dx}{dt} = 0.4$ m/s and $\frac{dy}{dt} = 9.6$ m/s.)

The area of the trapezoid is thus given by

$$A = 5y + \frac{1}{2}5x = 5y + 2.5x\,.$$

It follows that at any given instant

$$\frac{dA}{dt} = 5\frac{dy}{dt} + 2.5\frac{dx}{dt} = 5 \cdot 9.6 + 2.5 \cdot 0.4 = 48 + 1 = 49\ m^2/s\,.$$

Since this rate is constant, it is, in particular, how the area of the trapezoid is changing at the instant that Hretha crosses the finish line. ■

Part ♣. Do *one* (1) of **8** or **9**. *[15 = 1 × 15 each]*

8. Consider the power series $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{2^{n+1}} x^n$.

- **a.** Find the radius of convergence of this power series. *[10]*
- **b.** What function has this power series as its Taylor series at 0? *[5]*

Solution to **a**. We'll use the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)+1}{2^{(n+1)+1}}x^{n+1}}{\frac{n+1}{2^{n+1}}x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)+1}{2^{(n+1)+1}} \cdot \frac{2^{n+1}}{n+1} \cdot x\right| = \lim_{n\to\infty}\frac{n+2}{n+1}\cdot\frac{|x|}{2}$$
$$= \frac{|x|}{2}\lim_{n\to\infty}\frac{n+2}{n+1} = \frac{|x|}{2}\lim_{n\to\infty}\frac{n+2}{n+1}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} = \frac{|x|}{2}\lim_{n\to\infty}\frac{1+\frac{2}{n}}{1+\frac{1}{n}} = \frac{|x|}{2}\cdot\frac{1+0}{1+0} = \frac{|x|}{2}$$

It follows from the Ratio Test that the series converges (absolutely) when $\frac{|x|}{2} < 1$, *i.e.* when $|x| < 2$, and diverges when $\frac{|x|}{2} > 1$, *i.e.* when $|x| > 2$. This means that the radius of convergence of the given power series is $R = 2$. ■

Solution to **b**. Note that $\displaystyle\frac{d}{dx}\left(\frac{x^{n+1}}{2^{n+1}}\right) = \frac{n+1}{2^{n+1}}x^n$ for each $n \geq 0$. The series $\displaystyle\sum_{n=0}^{\infty}\frac{x^{n+1}}{2^{n+1}}$ is a geometric series with initial term $a = \frac{x}{2}$ and common ratio $r = \frac{x}{2}$, which is therefore equal to $\displaystyle\frac{a}{1-r} = \frac{\frac{x}{2}}{1-\frac{x}{2}} = \frac{x}{2-x}$ when it converges. It follows that

$$\sum_{n=0}^{\infty}\frac{n+1}{2^{n+1}}x^n = \sum_{n=0}^{\infty}\frac{d}{dx}\left(\frac{x^{n+1}}{2^{n+1}}\right) = \sum_{n=0}^{\infty}\frac{x^{n+1}}{2^{n+1}} = \frac{d}{dx}\frac{x}{2-x} = \frac{1(2-x)-x(-1)}{(2-x)^2} = \frac{2}{(2-x)^2}\,,$$

at least within its radius of convergence. Since a power series equal to a function must be that function's Taylor series, this means that $\displaystyle\sum_{n=0}^{\infty}\frac{n+1}{2^{n+1}}x^n$ is the Taylor series of $f(x) = \displaystyle\frac{2}{(2-x)^2}$. ■

9. Let $f(x) = x\sin(3x)$.

a. Find the Taylor series at 0 of $f(x)$. *[10]*

b. Determine the radius of convergence of this Taylor series. *[5]*

Solution to **a**. We'll do this indirectly, using the fact that $\sin(u) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} u^{2n+1}$. It follows that

$$f(x) = x\sin(3x) = x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(3x)^{2n+1}$$
$$= x \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n+1)!} x^{2n+2},$$

at least when the series converges. Since a power series equal to a function must be that function's Taylor series, this means that the Taylor series at 0 of $f(x) = x\sin(3x)$ is $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n+1)!} x^{2n+2}$. ■

Solution to **b**. The series obtained in the solution to **a** converges for all x. This can be shown pretty easily using the Ratio Test, but it can also be reasoned out by examining how it was obtained in that solution:

> The Taylor series at 0 for $\sin(u)$ is known (from, say, the class or the text) to converge for all u, so it must converge for $u = 3x$ no matter what the value of x is. Multiplying the series for $\sin(3x)$ by x is not going to change whether it converges or not. [Except, hypothetically, that a series that fails to converge may be forced to do so if you multiply through by $x = 0$.] Thus the series obtained in the solution to **a** must converge for all x. ■

[Total = 100]

Part ♠. Bonus problems! Do them (or not), if you feel like it.

0. Sketch the graph of $r = 1 - e^{-\theta}$ [polar coordinates!] for $\theta \geq 0$, and explain why it has the shape it does. *[2]*

−1. Write an original poem touching on calculus or mathematics in general. *[2]*

I the course was fun, at least a little.
Enjoy the rest of the summer!