Trent University

MATH 1100Y Test #1

Wednesday, 8 June, 2011

Time: 50 minutes

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Student Number: 01234567

Question	Mark
1	____
2	____
3	____
4	____
Total	____

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Find $\frac{dy}{dx}$ in any *three* (3) of **a**–**d**. *[9 = 3 × 3 each]*

a. $y = \left(x^2+1\right)^3$ **b.** $\ln(x+y) = 0$ **c.** $y = x^2 e^x$ **d.** $y = \dfrac{\tan(x)}{\sec(x)}$

SOLUTION TO **a**. Power and Chain Rules:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2+1\right)^3 = 3\left(x^2+1\right)^2 \cdot \frac{d}{dx}\left(x^2+1\right) = 3\left(x^2+1\right)^2(2x+0) = 6x\left(x^2+1\right)^2 \quad \square$$

SOLUTION I TO **b**. Solve for y first, then differentiate:

$$\begin{aligned} \ln(x+y) = 0 \quad &\implies \quad x+y = e^{\ln(x+y)} = e^0 = 1 \\ &\implies \quad y = 1-x \quad \implies \quad \frac{dy}{dx} = 0-1 = -1 \quad \square \end{aligned}$$

SOLUTION II TO **b**. Implicit differentiation:

$$\begin{aligned} \ln(x+y) = 0 \quad &\implies \quad \frac{d}{dx}\ln(x+y) = \frac{d}{dx}0 \quad \implies \quad \frac{1}{x+y}\cdot\frac{d}{dx}(x+y) = 0 \\ &\implies \quad \frac{1}{x+y}\cdot\left(1+\frac{dy}{dx}\right) = 0 \quad \implies \quad 1+\frac{dy}{dx} = (x+y)\cdot 0 = 0 \\ &\implies \quad \frac{dy}{dx} = 0-1 = -1 \quad \square \end{aligned}$$

SOLUTION TO **c**. Product Rule:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2e^x\right) = \left(\frac{d}{dx}x^2\right)\cdot e^x + x^2\cdot\left(\frac{d}{dx}e^x\right) = 2xe^x + x^2e^x = x(2+x)e^x \quad \square$$

SOLUTION I TO **d**. Simplify first, $y = \dfrac{\tan(x)}{\sec(x)} = \dfrac{\frac{\sin(x)}{\cos(x)}}{\frac{1}{\cos(x)}} = \dfrac{\sin(x)}{\cos(x)}\cdot\dfrac{\cos(x)}{1} = \sin(x)$, then differentiate, so $\dfrac{dy}{dx} = \dfrac{d}{dx}\sin(x) = \cos(x)$. $\square$

SOLUTION II TO **d**. Quotient Rule first, then simplify:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{\tan(x)}{\sec(x)}\right) = \frac{\left(\frac{d}{dx}\tan(x)\right)\cdot\sec(x) - \tan(x)\cdot\left(\frac{d}{dx}\sec(x)\right)}{\sec^2(x)} \\ &= \frac{\sec^2(x)\cdot\sec(x) - \tan(x)\cdot\sec(x)\tan(x)}{\sec^2(x)} = \frac{\sec^2(x)-\tan^2(x)}{\sec(x)} \\ &= \frac{\sec^2(x) - \left(\sec^2(x)-1\right)}{\sec(x)} = \frac{1}{\sec(x)} = \frac{1}{\frac{1}{\cos(x)}} = \cos(x) \quad \square \end{aligned}$$

2. Do any *two* (2) of **a**–**c**. *[10 = 2 × 5 each]*

a. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x \to 2} (x+1) = 3$.

b. Use the limit definition of the derivative to compute $f'(0)$ for $f(x) = x^3 + x$.

c. Compute $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$.

Solution to **a**. Suppose an $\varepsilon > 0$ is given. As usual, we attempt to reverse-engineer the required δ.

$$|(x+1) - 3| < \varepsilon \quad \iff \quad |x - 2| < \varepsilon$$

Since the step taken above is reversible, it follows that if we set $\delta = \varepsilon$, then whenever $|x-2| < \delta$, we will have $|(x+1) - 3| < \varepsilon$ also, as required.

Hence $\lim_{x \to 2} (x+1) = 3$ by the $\varepsilon - \delta$ definition of limits. □

Solution to **b**. Here goes:

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{\left(h^3 + h\right) - \left(0^3 + 0\right)}{h} \\ &= \lim_{h \to 0} \frac{h^3 + h}{h} = \lim_{h \to 0} \left(h^2 + 1\right) = 0^2 + 1 = 1 \quad \square \end{aligned}$$

Solution to **c**. Here goes:

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x+3)(x-3)}{x - 3} = \lim_{x \to 3} (x+3) = 3 + 3 = 6 \quad \square$$

3. Do any *two* (2) of **a**–**c**. *[12 = 2 × 6 each]*

a. Each side of a square is increasing at a rate of $3\,cm/s$. At what rate is the area of the square increasing at the instant that the sides are $6\,cm$ long?

b. $f(x) = e^{-1/x^2} = e^{-(x^{-2})}$ has a removable discontinuity at $x = 0$. What should the value of $f(0)$ be to make the function continuous at $x = 0$?

c. What is the smallest possible perimeter of a rectangle with area $36\,cm^2$?

SOLUTION TO **a**. Suppose the we denote the length of a side of the square by s, so its area will be $A = s^2$. We are given that $\frac{ds}{dt} = 3$ and we wish to know $\left.\frac{dA}{dt}\right|$ at the instant that $s = 6$. We differentiate A, plug in, and then solve. $\frac{dA}{dt} = \frac{d}{dt}s^2 = 2s \cdot \frac{ds}{dt}$, so when $s = 6$, we get $\frac{dA}{dt} = 2 \cdot 6 \cdot 3 = 36\,cm^2/s$. □

SOLUTION TO **b**. $f(x)$ being continuous at $x = 0$ amounts to having $f(0) = \lim_{x\to 0} f(x)$, so we need to compute this limit.

As $x \to 0$, $\frac{1}{x^2} \to +\infty$ (note that $x^2 > 0$ for all $x \neq 0$), and so $-\frac{1}{x^2} \to -\infty$. It follows that $\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{-1/x^2} = \lim_{t\to -\infty} e^t = 0$. Thus the value of $f(0)$ should be 0 to make $f(x)$ continuous at $x = 0$. □

SOLUTION TO **c**. Suppose a rectangle has height h and base b. Then its perimeter is $P = 2h + 2b$ and its area is $A = bh$. Note that both b and h need to be > 0 for any real rectangle with positive area.

In this case $A = bh = 36$, so $h = \frac{36}{b}$ and $P = 2\frac{36}{b} + 2b = \frac{72}{b} + 2b$, where $0 \leq b < \infty$. We first find the derivative, $\frac{dP}{db} = \frac{d}{db}\left(\frac{72}{b} + 2b\right) = -\frac{72}{b^2} + 2$, and then build the usual table. $\frac{dP}{db} = -\frac{72}{b^2} + 2 = 0$ exactly when $2b^2 = 72$, *i.e.* $b^2 = 36$, so that $b = 6$. (Recall that b must be > 0.) Similarly, $\frac{dP}{db} = -\frac{72}{b^2} + 2 > 0$ exactly when $2b^2 > 72$, *i.e.* $b^2 > 36$, so that $b > 6$, and $\frac{dP}{db} = -\frac{72}{b^2} + 2 < 0$ exactly when $2b^2 < 72$, *i.e.* $b^2 < 36$, so that $b < 6$. This gives the table:

b	$(0, 6)$	6	$(6, \infty)$
P	$\downarrow$	min	$\uparrow$
$\frac{dP}{db}$	$-$	0	$+$

It follows that P has its only minimum when $b = 6$, so the smallest possible perimeter of a rectangle of area $36\,cm^2$ is $P = \frac{72}{6} + 2 \cdot 6 = 12 + 12 = 24\,cm$. Note that this rectangle is the square with sides of length $6\,cm$. □

4. Let $f(x) = \sqrt{x^2+1}$. Find any and all intercepts, vertical and horizontal asymptotes, and maxima and minima of $f(x)$, and sketch its graph using this information. *[9]*

SOLUTION. *i. (Domain)* x^2+1 is defined, continuous, differentiable, and ≥ 1 for all x. Since $\sqrt{t}$ is defined, continuous, and differentiable when $t > 0$, it follows that $f(x) = \sqrt{x^2+1}$ is defined, continuous, and differentiable for all x.

ii. (Intercepts) $f(0) = \sqrt{0^2+1} = \sqrt{1} = 1$, so the y-intercept is the point $(0, 1)$. Since $\sqrt{x^2+1} \geq \sqrt{1} = 1 > 0$ for all x, there is no x such that $f(x) = 0$, *i.e.* $f(x)$ has no x-intercepts.

iii. (Vertical asymptotes) Since $f(x)$ is defined and continuous for all x, as noted in *i* above, it has no vertical asymptotes.

iv. (Horizontal asymptotes) To compute the relevant limits, observe that as $x \to \pm\infty$, $x^2+1 \to +\infty$, and hence $\sqrt{x^2+1} \to +\infty$. Since $\lim_{x\to+\infty} \sqrt{x^2+1} = +\infty = \lim_{x\to-\infty} \sqrt{x^2+1}$, $f(x) = \sqrt{x^2+1}$ has no horizontal asymptotes.

v. (Maxima & minima, etc.) Using the Chain and Power Rules,

$$f'(x) = \frac{d}{dx}\sqrt{x^2+1} = \frac{1}{2\sqrt{x^2+1}} \cdot \frac{d}{dx}\left(x^2+1\right) = \frac{1}{2\sqrt{x^2+1}} \cdot (2x+0) = \frac{x}{\sqrt{x^2+1}}\,.$$

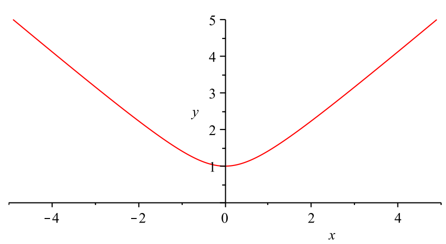
It follows that $f'(x) = 0$ if and only if $x = 0$. Moreover, since $\sqrt{x^2+1} \geq 1 > 0$ for all x, $f'(x)$ is < 0 or > 0 exactly when $x < 0$ or $x > 0$, respectively. Here is the usual table:

x	$(-\infty, 0)$	0	$(0, +\infty)$
$f(x)$	$\downarrow$	min	$\uparrow$
$f'(x)$	$+$	0	$-$

Thus $f(x)$ must have a minimum at the sole critical point of $x = 0$.

vi. (Graph) Cheating a bit and using `Maple`:

```
> plot(sqrt(x^2+1),x=-5..5,y=0..5);
```



$\square$

[Total = 40]

Bonus. Simplify $\cos(\arcsin(x))$ as much as you can. *[1]*

SOLUTION. $\cos(\arcsin(x)) = \sqrt{1-\sin^2(\arcsin(x))} = \sqrt{1-x^2}$ ought to do. $\square$